

QUADRATIC SPACES AND HOLOMORPHIC FRAMED VERTEX OPERATOR ALGEBRAS OF CENTRAL CHARGE 24

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ABSTRACT. In 1993, Schellekens [Sc93] obtained a list of possible 71 Lie algebras of holomorphic vertex operator algebras with central charge 24. However, not all cases are known to exist. The aim of this article is to construct new holomorphic vertex operator algebras using the theory of framed vertex operator algebras and to determine the Lie algebra structures of their weight one subspaces. In particular, we study holomorphic framed vertex operator algebras associated to subcodes of the triply even codes $\text{RM}(1, 4)^3$ and $\text{RM}(1, 4) \oplus \mathcal{D}(d_{16}^+)$ of length 48. These vertex operator algebras correspond to the holomorphic simple current extensions of the lattice type vertex operator algebras $(V_{\sqrt{2}E_8}^+)^{\otimes 3}$ and $V_{\sqrt{2}E_8}^+ \otimes V_{\sqrt{2}D_{16}^+}^+$. We determine such extensions using a quadratic space structure on the set of all irreducible modules $R(W)$ of W when $W = (V_{\sqrt{2}E_8}^+)^{\otimes 3}$ or $V_{\sqrt{2}E_8}^+ \otimes V_{\sqrt{2}D_{16}^+}^+$ [Sh04, Sh11]. As our main results, we construct seven new holomorphic vertex operator algebras of central charge 24 in Schellekens' list and obtain a complete list of all Lie algebra structures associated to the weight one subspaces of holomorphic framed vertex operator algebras of central charge 24.

INTRODUCTION

The classification of even unimodular lattices of rank 24 is one of fundamental results in lattice theory; there are exactly 24 such lattices and each lattice is uniquely determined by its root system – the set of norm 2 vectors. Since there are many analogies between lattices and vertex operator algebras (VOAs), it is natural to consider the corresponding classification problem for holomorphic VOAs of central charge 24. In 1993 Schellekens [Sc93] obtained a list of possible 71 Lie algebras of holomorphic vertex operator algebras with central charge 24. However, not all cases are constructed explicitly and known to exist. In fact, only the cases for VOAs associated to even unimodular lattices and their \mathbb{Z}_2 -orbifolds are well-studied ([FLM88, DGM96]).

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Framed VOA is another class of well-studied VOAs ([DGH98, Mi04, LY08]). Roughly speaking, a framed VOA is a simple VOA which contains a full subalgebra T , called a Virasoro frame, isomorphic to a tensor product of copies of the Virasoro VOA $L(1/2, 0)$. Such a VOA is rational and C_2 -cofinite, and its structure is mainly determined by certain combinatorial objects. Therefore, it is natural to consider the classification of holomorphic framed VOAs of central charge 24.

It was shown in [LY08] that a binary code D of length $16k$ can be realized as the $1/16$ -code of a holomorphic framed VOA of central charge $8k$ if and only if D is triple even (i.e., $\text{wt}(\alpha) \equiv 0 \pmod{8}$ for all $\alpha \in D$) and the all-one vector $(1, \dots, 1) \in D$. Therefore, the classification of holomorphic framed VOAs of central charge 24 is almost equivalent to the problem of classifying all triply even codes of length 48 and the study of possible VOA structures associated to each triply even code.

In [BM], triply even codes of length 48 are classified: any triply even code of length 48 is a subcode of one of the following:

- (1) an extended doubling $\mathcal{D}(E)$ for some doubly even code E of length 24 (see Definition 2.12);
- (2) the 9-dimensional exceptional triply even code D^{ex} of length 48;
- (3) the direct sum of three copies of the Reed-Muller code $\text{RM}(1, 4)$;
- (4) the direct sum of $\text{RM}(1, 4)$ and $\mathcal{D}(d_{16}^+)$, where d_{16}^+ is the unique indecomposable doubly even self-dual code of length 16.

It was shown in [La11] that if the $1/16$ -code is a subcode of an extended doubling (Case 1), then the framed VOA is isomorphic to a lattice VOA or its \mathbb{Z}_2 -orbifold. Moreover, holomorphic framed VOAs associated to subcodes of D^{ex} (Case 2) have been constructed and studied in [La11]. The Lie algebras associated to their weight one subspaces are also determined. In particular, 10 new holomorphic framed VOAs are found mathematically. In order to complete the classification of holomorphic framed VOAs, it is very important to construct and study the holomorphic framed VOAs associated to the subcodes in Cases 3 and 4.

In this article, we will study the framed VOAs associated to the triply even codes isomorphic to subcodes of $\text{RM}(1, 4)^3$ and $\text{RM}(1, 4) \oplus \mathcal{D}(d_{16}^+)$. These VOAs correspond to the holomorphic simple current extensions of $(V_{\sqrt{2}E_8}^+)^{\otimes 3}$ and $V_{\sqrt{2}E_8}^+ \otimes V_{\sqrt{2}D_{16}^+}^+$. It was proved in [Sh04] that the set of all irreducible modules $R(V_{\sqrt{2}E_8}^+)$ and $R(V_{\sqrt{2}D_{16}^+}^+)$ of $V_{\sqrt{2}E_8}^+$ and $V_{\sqrt{2}D_{16}^+}^+$ have the structures of some quadratic spaces and $\text{Aut}(V_{\sqrt{2}E_8}^+)$ and $\text{Aut}(V_{\sqrt{2}D_{16}^+}^+)$ act naturally on $R(V_{\sqrt{2}E_8}^+)$ and $R(V_{\sqrt{2}D_{16}^+}^+)$, respectively. By using these quadratic spaces,

we determine all holomorphic extensions of $(V_{\sqrt{2}E_8}^+)^{\otimes 3}$ and $V_{\sqrt{2}E_8}^+ \otimes V_{\sqrt{2}D_{16}^+}^+$, up to conjugation, and compute the associated Lie algebra structures of the corresponding weight one spaces. As a consequence, we construct seven new holomorphic framed VOAs having Lie algebras $D_{4,4}(A_{2,2})^4$, $C_{4,2}(A_{4,2})^2$, $E_{6,2}C_{5,1}A_{5,1}$, $C_{8,1}(F_{4,1})^2$, $E_{7,2}B_{5,1}F_{4,1}$, $E_{8,2}B_{8,1}$ and $A_{8,2}F_{4,2}$. In addition, we obtain a complete list of all Lie algebra structures for the weight one subspaces of holomorphic framed VOAs of central charge 24. The main result is as follows.

Theorem 0.1. *Let V be a holomorphic framed VOA of central charge 24. Then one of the following holds:*

- (1) V is isomorphic to a lattice VOA V_N or its \mathbb{Z}_2 -orbifold \tilde{V}_N for some even unimodular lattice N ;
- (2) the weight one subspace V_1 is isomorphic to one of the Lie algebras in Table 1.

Moreover, for each Lie algebra \mathfrak{L} in Table 1, there exists a holomorphic framed VOA U whose weight one subspace U_1 is isomorphic to \mathfrak{L} .

The organization of this article is as follows. In Section 1, we recall some basic facts about quadratic spaces and orthogonal groups. We also review the notions of VOAs, modules and intertwining operators. In Section 2, the notion and several important properties of framed VOAs will be reviewed. We will also recall the classification of triply even codes of length 48 from [BM]. In Section 3, the representation theory of the lattice type VOAs $V_{\sqrt{2}E_8}^+$ and $V_{\sqrt{2}D_{16}^+}^+$ is reviewed. In particular, the quadratic spaces associated to the set of their irreducible modules are discussed. In Section 4, we will study the holomorphic simple current extensions of $(V_{\sqrt{2}E_8}^+)^{\otimes 3}$. The main result is a complete description of all maximal totally singular subspaces of $R(V_{\sqrt{2}E_8}^+)^{\otimes 3}$, up to the action of $\text{Aut}((V_{\sqrt{2}E_8}^+)^{\otimes 3})$. The Lie algebra structures of the weight one subspaces of the corresponding framed VOAs associated to these quadratic spaces will also be computed. In Section 5, we study the holomorphic simple current extensions of $V_{\sqrt{2}E_8}^+ \otimes V_{\sqrt{2}D_{16}^+}^+$. We also obtain a complete classification of all maximal totally singular subspaces of $R(V_{\sqrt{2}E_8}^+) \oplus R(V_{\sqrt{2}D_{16}^+}^+)$, up to the action of $\text{Aut}(V_{\sqrt{2}E_8}^+ \otimes V_{\sqrt{2}D_{16}^+}^+)$. Again, the Lie algebras associated to the weight one subspaces of the corresponding framed VOAs will be computed.

TABLE 1. Lie algebras for holomorphic framed VOAs of $c = 24$

No. in [Sc93]	Dimension	Lie algebra
7	48	$(A_{3,4})^3 A_{1,2}$
10	48	$D_{5,8} A_{1,2}$
13	60	$D_{4,4} (A_{2,2})^4$
18	72	$A_{7,4} (A_{1,1})^3$
19	72	$D_{5,4} C_{3,2} (A_{1,1})^2$
22	84	$C_{4,2} (A_{4,2})^2$
26	96	$(A_{5,2})^2 C_{2,1} (A_{2,1})^2$
33	120	$A_{7,2} (C_{3,1})^2 A_{3,1}$
35	120	$C_{7,2} A_{3,1}$
36	132	$A_{8,2} F_{4,2}$
40	144	$A_{9,2} A_{4,1} B_{3,1}$
44	168	$E_{6,2} C_{5,1} A_{5,1}$
48	192	$(C_{6,1})^2 B_{4,1}$
52	240	$C_{8,1} (F_{4,1})^2$
53	240	$E_{7,2} B_{5,1} F_{4,1}$
56	288	$C_{10,1} B_{6,1}$
62	384	$E_{8,2} B_{8,1}$

1. PRELIMINARY

Notations

\langle , \rangle	The symplectic form on a quadratic space (R, q) or on (R^k, q^k) .
\boxtimes	The fusion rules for a VOA.
\perp	The orthogonal direct sum of subspaces in a quadratic space.
A^\perp	The orthogonal complement of a subspace A in a quadratic space.
D_{16}^+	The indecomposable even unimodular lattice of rank 16, whose root lattice is D_{16} .
E_8	The E_8 root lattice, even unimodular lattice of rank 8.
$g \circ M$	The conjugate of a module M for a VOA by an automorphism g .
$g \circ [M]$	The isomorphism class of $g \circ M$.
$[M]$	The isomorphism class of a module M for a VOA.
$N(L)$	The even unimodular lattice of rank 24 whose root lattice is isomorphic to L .
ρ_i	The i -th coordinate projection of a direct sum of quadratic spaces.
q_V	The quadratic form on $R(V)$ defined by $q_V([M]) = 0$ and 1 if M is \mathbb{Z} -graded and $(\mathbb{Z} + 1/2)$ -graded, respectively.
$O(R, q)$	The orthogonal group of (R, q) .
Sym_n	The symmetric group of degree n .
\mathcal{S}	A maximal totally singular subspace of R^k or $R(V)^k$.
$\mathcal{S}(m, k_1, k_2, \varepsilon)$	The maximal totally singular subspace of R^3 defined in Theorem 4.6.
$\mathcal{S}(m, k_1, k_2)$	The maximal totally singular subspace of R^3 defined in Theorem 4.8.
$S(R)$	The set of singular vectors in R
$S(R)^\times$	The set of non-zero singular vectors in R
$\overline{S(R)}$	The set of non-singular vectors in R
$\text{Stab}_G(A)$	The setwise stabilizer of A in a group G .
$\text{Stab}^{\text{pt}}_G(A)$	The pointwise stabilizer of A in a group G .
(R, q)	A non-singular quadratic space R of plus type with quadratic form q over \mathbb{F}_2 .
(R^k, q^k)	The orthogonal direct sum of k copies of (R, q) .
$R(V)$	The set of all isomorphism classes of irreducible modules for a VOA V .
V	A simple rational C_2 -cofinite self-dual VOA of CFT type, or $V = V_{\sqrt{2}E_8}^+$.
V_L	The lattice VOA associated with even lattice L .
V_L^+	The fixed point subVOA of V_L with respect to a lift of the -1 -isometry of L .
\tilde{V}_L	\mathbb{Z}_2 -orbifold of V_L .
V^k	The tensor product of k copies of a VOA V .
$\mathfrak{V}(\mathcal{S})$	The holomorphic VOA associated to a maximal totally singular subspace \mathcal{S} .
$\mathfrak{V}(\mathcal{T})$	The module associated to a subset \mathcal{T} of $R(V)^k$ or $R(X) \oplus R(V)$.
X	$X = V_{\sqrt{2}D_{16}^+}^+$.

1.1. Quadratic spaces and orthogonal groups. Let us recall fundamental facts on quadratic spaces over \mathbb{F}_2 and orthogonal groups (cf. [Ch97]).

Let R be a $2m$ -dimensional vector space over \mathbb{F}_2 . A form $\langle \cdot, \cdot \rangle : R \times R \rightarrow \mathbb{F}_2$ is said to be *symplectic* if it is a symmetric bilinear form with $\langle a, a \rangle = 0$ for all $a \in R$. A map $q : R \rightarrow \mathbb{F}_2$ is called a *quadratic form* associated to $\langle \cdot, \cdot \rangle$ if $\langle a, b \rangle = q(a+b) + q(a) + q(b)$ for all $a, b \in R$. Let q be a quadratic form. Then the pair (R, q) is called a *quadratic space*, and it is said to be *non-singular* if $R^\perp = \{a \in R \mid \langle a, R \rangle = 0\} = 0$. A vector $a \in R$ is said to be *singular* and *non-singular* if $q(a) = 0$ and $q(a) = 1$, respectively. Let $S(R)$, $S(R)^\times$, and $\overline{S(R)}$ denote the sets of all singular vectors in R , of all non-zero singular vectors in R , and of all non-singular vectors in R , respectively. A subspace S of R is said to be *totally singular* if any vector in S is singular. A quadratic form q is said to be of *plus type* and of *minus type* if the dimension of a maximal totally singular subspace of (R, q) is m and $m - 1$, respectively. Let $O(R, q)$ denote the orthogonal group of (R, q) , the subgroup of $\text{GL}(R)$ preserving q . The following lemmas are well-known.

Lemma 1.1. *Let (R, q) be a non-singular $2m$ -dimensional quadratic space of ε type over \mathbb{F}_2 , where $\varepsilon \in \{\pm\}$.*

- (1) *If $\varepsilon = +$ then R has $2^{2m-1} + 2^{m-1} - 1$ non-zero singular vectors and $2^{2m-1} - 2^{m-1}$ non-singular vectors.*
- (2) *If $\varepsilon = -$ then R has $2^{2m-1} - 2^{m-1} - 1$ non-zero singular vectors and $2^{2m-1} + 2^{m-1}$ non-singular vectors.*

Lemma 1.2. (cf. [Ch97]) *Let (R, q) be a non-singular $2m$ -dimensional quadratic space of plus type over \mathbb{F}_2 . Let $k \in \{1, 2, \dots, m\}$.*

- (1) *The orthogonal group $O(R, q)$ is transitive on the set of all k -dimensional totally singular subspaces of R .*
- (2) *The stabilizer of a k -dimensional totally singular subspace in $O(R, q)$ has the shape $2^{\binom{k}{2} + k(2m-2k)} \cdot (\text{SL}_k(2) \times O^+(2m-2k, 2))$.*
- (3) *Let U be a subspace of R such that $\langle U, U \rangle = 0$ and $\dim U \geq 2$. Then U contains a non-zero singular vector.*

1.2. Vertex operator algebras. Throughout this article, all VOAs are defined over the field \mathbb{C} of complex numbers unless otherwise stated. We recall the notions of vertex operator algebras (VOAs) and modules from [Bo86, FLM88, FHL93].

A *vertex operator algebra* (VOA) $(V, Y, \mathbf{1}, \omega)$ is a $\mathbb{Z}_{\geq 0}$ -graded vector space $V = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} V_m$ equipped with a linear map

$$Y(a, z) = \sum_{i \in \mathbb{Z}} a_{(i)} z^{-i-1} \in (\text{End}(V))[[z, z^{-1}]], \quad a \in V$$

and the *vacuum vector* $\mathbf{1}$ and the *conformal element* ω satisfying a number of conditions ([Bo86, FLM88]). We often denote it by V or (V, Y) .

Two VOAs $(V, Y, \mathbf{1}, \omega)$ and $(V', Y', \mathbf{1}', \omega')$ are said to be *isomorphic* if there exists a linear isomorphism g from V to V' such that

$$g\omega = \omega' \quad \text{and} \quad gY(v, z) = Y'(gv, z)g \quad \text{for all } v \in V.$$

When $V = V'$, such a linear isomorphism is called an *automorphism*. The group of all automorphisms of V is called the *automorphism group* of V and is denoted by $\text{Aut}(V)$.

A *vertex operator subalgebra* (or a *subVOA*) is a graded subspace of V which has a structure of a VOA such that the operations and its grading agree with the restriction of those of V and that they share the vacuum vector. When they also share the conformal element, we will call it a *full subVOA*.

An (ordinary) module (M, Y_M) for a VOA V is a \mathbb{C} -graded vector space $M = \bigoplus_{m \in \mathbb{C}} M_m$ equipped with a linear map

$$Y_M(a, z) = \sum_{i \in \mathbb{Z}} a_{(i)} z^{-i-1} \in (\text{End}(M))[[z, z^{-1}]], \quad a \in V$$

satisfying a number of conditions ([FHL93]). We often denote it by M and its isomorphism class by $[M]$. The *weight* of a homogeneous vector $v \in M_k$ is k . If M is irreducible then $M = \bigoplus_{m \in \mathbb{Z}_{\geq 0}} M_{h+m}$ ($h \in \mathbb{C}, M_h \neq 0$), where h is the *lowest weight* of M .

A VOA V is said to be of *CFT type* if $V_0 = \mathbb{C}\mathbf{1}$, is said to be *rational* if any module is completely reducible, and is said to be *C_2 -cofinite* if $V/\text{Span}_{\mathbb{C}}\{a_{(-2)}b \mid a, b \in V\}$ is finite-dimensional. Note that any module is ordinary if V is C_2 -cofinite. A VOA is said to be *holomorphic* if it is the only irreducible module up to isomorphism. A module M is said to be *self-dual* if its contragredient module (cf. [FHL93]) is isomorphic to itself. Let $R(V)$ denote the set of all isomorphism classes of irreducible V -modules. Note that if V is rational or C_2 -cofinite then $|R(V)| < \infty$.

Let M_a, M_b, M_c be modules for a simple rational C_2 -cofinite VOA V . An intertwining operator of type $M_a \times M_b \rightarrow M_c$ is a linear map $M_a \mapsto (\text{Hom}(M_b, M_c))\{z\}$ satisfying a number of conditions ([FHL93]). Let $N_{M_a, M_b}^{M_c}$ denote the dimension of the space of all intertwining operators of type $M_a \times M_b \rightarrow M_c$, which is called the *fusion rule*. Since V is C_2 -cofinite, the fusion rules are finite ([ABD04]). By the definition of the fusion rules, $N_{M_a, M_b}^{M_c} = N_{M'_a, M'_b}^{M'_c}$ if $M_x \cong M'_x$ as V -modules for all $x = a, b, c$. Hence, the fusion rules

are determined by the isomorphism classes of V -modules. For convenience, we use the following expression

$$[M_a] \boxtimes [M_b] = \bigoplus_{[M] \in R(V)} N_{M_a, M_b}^M [M],$$

which is also called the fusion rule.

Let $M = (M, Y_M)$ be a module for a VOA V . For $g \in \text{Aut}(V)$, let $g \circ M = (M, Y_{g \circ M})$ denote the V -module defined by $Y_{g \circ M}(v, z) = Y_M(g^{-1}v, z)$. If $M \cong M'$ as V -modules, then $g \circ M \cong g \circ M'$ as V -modules. Hence we use the notation $g \circ [M]$ for the isomorphism class of $g \circ M$. If M is irreducible, then so is $g \circ M$. Hence $\text{Aut}(V)$ acts on $R(V)$. We often identify modules of a VOA with their respective isomorphism classes and use the same notation for the conjugates.

The theory of tensor products of VOAs was established in [FHL93]. For a positive integer k , let V^k denote the tensor product of k copies of V . Later, we use the following lemma.

Lemma 1.3. ([FHL93, Section 4.7], [DMZ94]) *Let V be a simple rational C_2 -cofinite VOA of CFT type. Then*

$$R(V^k) = \{\otimes_{i=1}^k W_i \mid W_i \in R(V)\},$$

and for $\otimes_{i=1}^k W_{i,a}, \otimes_{i=1}^k W_{i,b} \in R(V^k)$, the following fusion rule holds:

$$(\otimes_{i=1}^k W_{i,a}) \boxtimes (\otimes_{i=1}^k W_{i,b}) = \otimes_{i=1}^k (W_{i,a} \boxtimes W_{i,b}).$$

Let $V(0)$ be a simple VOA. An irreducible $V(0)$ -module M_a is called a *simple current module* if for any irreducible $V(0)$ -module M_b , there exists a unique irreducible $V(0)$ -module M_c satisfying the fusion rule $[M_a] \boxtimes [M_b] = [M_c]$. A simple VOA V is called a *simple current extension* of $V(0)$ graded by a finite abelian group E if V is the direct sum of non-isomorphic irreducible simple current $V(0)$ -modules $\{V(\alpha) \mid \alpha \in E\}$ and the fusion rule $[V(\alpha)] \boxtimes [V(\beta)] = [V(\alpha + \beta)]$ holds for all $\alpha, \beta \in E$.

Lemma 1.4. [SY03, Lemma 3.14] *Let $V = \oplus_{\alpha \in E} V(\alpha)$ be a simple current extension of a simple VOA $V(0)$ graded by a finite abelian group E . Let g be an automorphism of $V(0)$. Then there exists a simple current extension of $V(0)$ such that it is isomorphic to V as VOAs and is isomorphic to $\oplus_{\alpha \in E} g \circ V(\alpha)$ as $V(0)$ -modules.*

1.3. Lattice VOAs and \mathbb{Z}_2 -orbifolds. Let L be an even unimodular lattice and let V_L be the lattice VOA associated with L ([Bo86, FLM88]). Then V_L is holomorphic ([Do93]). Let $\theta \in \text{Aut}(V_L)$ be a lift of $-1 \in \text{Aut}(L)$ and let V_L^+ denote the subVOA of V_L consisting

of vectors in V_L fixed by θ . Let V_L^T be a unique θ -twisted module for V_L and $V_L^{T,+}$ the irreducible V_L^+ -submodule of V_L^T with integral weights. Let

$$\tilde{V}_L = V_L^+ \oplus V_L^{T,+}.$$

It is known that \tilde{V}_L has a unique holomorphic VOA structure by extending its V_L^+ -module structure (see [FLM88, DGM96]). The VOA \tilde{V}_L is often called the \mathbb{Z}_2 -orbifold of V_L .

Remark 1.5. Assume that the rank of L is 24. Then any even unimodular lattice L has an orthogonal basis of norm 4 ([HK00]). Hence both V_L and \tilde{V}_L are framed ([DMZ94]).

2. FRAMED VERTEX OPERATOR ALGEBRAS

In this section, we review the notion of framed VOAs from [DGH98, Mi04].

Let $\text{Vir} = \bigoplus_{n \in \mathbb{Z}} \mathbb{C}L_n \oplus \mathbb{C}\mathbf{c}$ be the Virasoro algebra. That means the L_n satisfy the commutator relations:

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{1}{12}(m^3 - m)\delta_{m+n,0}\mathbf{c}, \quad \text{and} \quad [L_m, \mathbf{c}] = 0.$$

For any $c, h \in \mathbb{C}$, we will denote by $L(c, h)$ the irreducible highest weight module of Vir with central charge c and highest weight h . It is shown by [FZ92] that $L(c, 0)$ has a natural VOA structure. We call it the *simple Virasoro VOA* with central charge c .

Definition 2.1. Let $V = \bigoplus_{n=0}^{\infty} V_n$ be a VOA. An element $e \in V_2$ is called an *Ising vector* if the subalgebra $\text{Vir}(e)$ generated by e is isomorphic to $L(1/2, 0)$ and e is the conformal element of $\text{Vir}(e)$. Two Ising vectors $u, v \in V$ are said to be *orthogonal* if $[Y(u, z_1), Y(v, z_2)] = 0$.

Remark 2.2. It is well-known that $L(1/2, 0)$ is rational, i.e., all $L(1/2, 0)$ -modules are completely reducible, and has only three inequivalent irreducible modules $L(1/2, 0)$, $L(1/2, 1/2)$ and $L(1/2, 1/16)$. The fusion rules of $L(1/2, 0)$ -modules are computed in [DMZ94]:

$$(2.1) \quad \begin{aligned} L(1/2, 1/2) \boxtimes L(1/2, 1/2) &= L(1/2, 0), & L(1/2, 1/2) \boxtimes L(1/2, 1/16) &= L(1/2, 1/16), \\ L(1/2, 1/16) \boxtimes L(1/2, 1/16) &= L(1/2, 0) \oplus L(1/2, 1/2). \end{aligned}$$

Definition 2.3. ([DGH98]) A simple VOA V is said to be *framed* if there exists a set $\{e^1, \dots, e^n\}$ of mutually orthogonal Ising vectors of V such that their sum $e^1 + \dots + e^n$ is equal to the conformal element of V . The subVOA T_n generated by e^1, \dots, e^n is thus isomorphic to $L(1/2, 0)^{\otimes n}$ and is called a *Virasoro frame* or simply a *frame* of V .

2.1. Structure codes. Given a framed VOA V with a frame T_n , one can associate two binary codes C and D of length n to V and T_n as follows:

Since $T_n = L(1/2, 0)^{\otimes n}$ is rational, V is a completely reducible T_n -module. That is,

$$V \cong \bigoplus_{h_i \in \{0, \frac{1}{2}, \frac{1}{16}\}} m_{h_1, \dots, h_n} L(1/2, h_1) \otimes \cdots \otimes L(1/2, h_n),$$

where the nonnegative integer m_{h_1, \dots, h_n} is the multiplicity of $L(1/2, h_1) \otimes \cdots \otimes L(1/2, h_n)$ in V . Then all the multiplicities are finite. It was also shown in [DMZ94] that m_{h_1, \dots, h_n} is at most 1 if all h_i are different from $1/16$.

Definition 2.4. Let $U \cong L(1/2, h_1) \otimes \cdots \otimes L(1/2, h_n)$ be an irreducible module for T_n . We define the $1/16$ -word (or τ -word) $\tau(U)$ of U as the binary word $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}_2^n$ such that

$$(2.2) \quad \beta_i = \begin{cases} 0 & \text{if } h_i = 0 \text{ or } 1/2, \\ 1 & \text{if } h_i = 1/16. \end{cases}$$

For any $\beta \in \mathbb{Z}_2^n$, denote by V^β the sum of all irreducible submodules U of V such that $\tau(U) = \beta$.

Definition 2.5. Define $D := \{\beta \in \mathbb{Z}_2^n \mid V^\beta \neq 0\}$. Then D becomes a binary code of length n and V can be written as a sum

$$V = \bigoplus_{\beta \in D} V^\beta.$$

For any $c = (c_1, \dots, c_n) \in \mathbb{Z}_2^n$, denote $M_c = m_{h_1, \dots, h_n} L(1/2, h_1) \otimes \cdots \otimes L(1/2, h_n)$ where $h_i = 1/2$ if $c_i = 1$ and $h_i = 0$ elsewhere. Note that $m_{h_1, \dots, h_n} \leq 1$ since $h_i \neq 1/16$.

Definition 2.6. Define $C := \{c \in \mathbb{Z}_2^n \mid M_c \neq 0\}$. Then C also forms a binary code and $V^0 = \bigoplus_{c \in C} M_c$.

Remark 2.7. The VOA V^0 is often called the *code VOA* associated to C and is denoted by M_C ([Mi96]).

Summarizing, there exists a pair of binary codes (C, D) such that

$$V = \bigoplus_{\beta \in D} V^\beta \quad \text{and} \quad V^0 = \bigoplus_{c \in C} M_c.$$

The codes (C, D) are called the *structure codes* of a framed VOA V associated to the frame T_n . We also call the code D the $\frac{1}{16}$ -code and the code C the $\frac{1}{2}$ -code of V with respect to T_n . Note also that all V^β , $\beta \in D$, are irreducible V^0 -modules.

Since V is a VOA, its weights are integers and we have the lemma.

Lemma 2.8. (1) *The code D is triply even, i.e., $\text{wt}(\alpha) \equiv 0 \pmod{8}$ for all $\alpha \in D$.*
(2) *The code C is even.*

The following theorem is also well-known (cf. [DGH98, Theorem 2.9] and [Mi04, Theorem 6.1]):

Theorem 2.9. *Let V be a framed VOA with structure codes (C, D) . Then, V is holomorphic if and only if $C = D^\perp$.*

In [LY08], the structure of a general framed VOA has been studied in detail. In particular, the following is established (see [LY08, Theorem 10]).

Theorem 2.10. *Let D be a linear binary code of length $16k, k \in \mathbb{Z}_{>0}$. Then D can be realized as the $\frac{1}{16}$ -code of a holomorphic framed VOA of central charge $8k$ if and only if (1) D is triply even and (2) the all-one vector $(1^{16k}) \in D$.*

By the theorem above, the classification of the $\frac{1}{16}$ -codes for holomorphic framed VOAs is equivalent to the classification of triply even codes of length $16k$.

2.2. Triply even codes of length 48. Triply even binary codes of length 48 have been classified recently by Betsumiya and Munemasa [BM]. In this subsection, we will recall their result.

Definition 2.11. Let n be a positive integer. We define two linear maps $d : \mathbb{Z}_2^n \rightarrow \mathbb{Z}_2^{2n}$, $\ell : \mathbb{Z}_2^n \rightarrow \mathbb{Z}_2^{2n}$ such that

$$(2.3) \quad \begin{aligned} d(a_1, a_2, \dots, a_n) &= (a_1, a_1, a_2, a_2, \dots, a_n, a_n), \\ \ell(a_1, a_2, \dots, a_n) &= (a_1, 0, a_2, 0, \dots, a_n, 0), \end{aligned}$$

for any $(a_1, a_2, \dots, a_n) \in \mathbb{Z}_2^n$.

Definition 2.12. Let E be a binary code of length n . We will define

$$\mathcal{D}(E) = \text{Span}_{\mathbb{Z}_2} \{d(E), \ell(1^n)\}$$

to be the code generated by $d(E)$ and $\ell(1^n)$. We call the binary code $\mathcal{D}(E)$ of length $2n$ the *extended doubling* (or simply the *doubling*) of E .

Lemma 2.13. *If E is a k -dimensional doubly even binary code of length $8n$, then the doubling $\mathcal{D}(E)$ is a $(k+1)$ -dimensional triply even code of length $16n$.*

Notation 2.14. For any positive integer n , let \mathcal{E}_n be the subcode of \mathbb{Z}_2^n consisting of all even codewords. Note that $\mathcal{D}(E)^\perp$ contains $d(\mathcal{E}_n)$ for any binary code E .

Notation 2.15. Let $k \geq 2$. We denote by d_{2k} the doubly even binary code of length $2k$ generated by

$$\begin{pmatrix} 1 & 1 & 1 & 1 & & & & \\ & & 1 & 1 & 1 & 1 & & \\ & & & & \ddots & \ddots & & \\ & & & & & 1 & 1 & 1 & 1 \end{pmatrix}.$$

We also denote the doubly even binary codes generated by

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

by e_7 and e_8 , respectively.

Notation 2.16. Let a_1, \dots, a_k be doubly even binary codes generated by its weight 4 elements. We will use $(a_1 \cdots a_k)^+$ to denote the doubly even self-dual code whose weight 4 elements generate a subcode $a_1 \oplus \cdots \oplus a_k$. We also use g_{24} to denote the binary Golay code of length 24. Note that g_{24} has no element of weight 4.

Recently, Betsumiya and Munemasa [BM] have classified all maximal triply even binary codes of length 48. Their main result is as follows.

Theorem 2.17 (cf. [BM]). *Let D be a triply even code of length 48. Then D is isomorphic to a subcode of one of the following codes:*

- (1) $\mathcal{D}(g_{24}), \mathcal{D}((d_{10}e_7^2)^+), \mathcal{D}(d_{24}^+), \mathcal{D}((d_{12}^2)^+), \mathcal{D}((d_4^6)^+), \mathcal{D}((d_6^4)^+), \mathcal{D}((d_8^3)^+),$
- (2) $\mathcal{D}(e_8)^{\oplus 3},$
- (3) $\mathcal{D}(e_8) \oplus \mathcal{D}(d_{16}^+),$
- (4) $D^{\text{ex}},$ a maximal triply even code of dimension 9.

Remark 2.18. Note that there are exactly nine non-equivalent doubly even self-dual codes of length 24. The codes $g_{24}, (d_{10}e_7^2)^+, d_{24}^+, (d_{12}^2)^+, (d_4^6)^+, (d_6^4)^+$ and $(d_8^3)^+$ are indecomposable while e_8^3 and $e_8 \oplus d_{16}^+$ are decomposable.

By the theorem above, we know that most triply even codes of length 48 are contained in some extended doublings. The following theorem characterizes all holomorphic framed VOAs with the $1/16$ -code D contained in an extended doubling.

Theorem 2.19. [La11, Theorem 3.9] *Let $V = \oplus_{\beta \in D} V^\beta$ be a holomorphic framed VOA with the $\frac{1}{16}$ -code D . Suppose that D can be embedded into a doubling $\mathcal{D}(E)$ for some*

doubly even code E . Then there is an even unimodular lattice N such that $V \cong V_N$ or \tilde{V}_N .

Because of Theorem 2.19, we only concentrate on triply even codes which are not contained in any doubling. Such codes must be subcodes of $\mathcal{D}(e_8)^{\oplus 3}$, $\mathcal{D}(e_8) \oplus \mathcal{D}(d_{16}^+)$ or D^{ex} . In [La11], holomorphic VOAs associated to subcodes of D^{ex} has been studied and the following theorem is proved.

Theorem 2.20. [La11, Theorem 6.78 and Table 1] *Let $V = \oplus_{\beta \in D} V^\beta$ be a holomorphic framed VOA of central charge 24 with the $\frac{1}{16}$ -code D . Assume that D is a subcode of D^{ex} . Then either*

- (1) *V is isomorphic to $V \cong V_N$ or \tilde{V}_N for some even unimodular lattice N ; or*
- (2) *the weight one subspace V_1 is isomorphic to one of the Lie algebra listed in Table 2.*

Moreover, for each Lie algebra \mathfrak{L} in Table 2, there is a holomorphic framed VOA U such that $U_1 \cong \mathfrak{L}$.

TABLE 2. Lie algebras associated to D^{ex}

No. in [Sc93]	Dimension	Lie algebra
7	48	$(A_{3,4})^3 A_{1,2}$
10	48	$D_{5,8} A_{1,2}$
18	72	$A_{7,4} (A_{1,1})^3$
19	72	$D_{5,4} C_{3,2} (A_{1,1})^2$
26	96	$(A_{5,2})^2 C_{2,1} (A_{2,1})^2$
33	120	$A_{7,2} (C_{3,1})^2 A_{3,1}$
35	120	$C_{7,2} A_{3,1}$
40	144	$A_{9,2} A_{4,1} B_{3,1}$
48	192	$(C_{6,1})^2 B_{4,1}$
56	288	$C_{10,1} B_{6,1}$

Because of Theorems 2.19 and 2.20, we will only study holomorphic framed VOAs associated to subcodes of $\mathcal{D}(e_8)^{\oplus 3}$ and $\mathcal{D}(e_8) \oplus \mathcal{D}(d_{16}^+)$ in the remaining of this article.

First, we note that $\mathcal{D}(e_8) \cong \text{RM}(1, 4)$ and $\text{RM}(1, 4)^\perp = \text{RM}(2, 4)$, where $\text{RM}(k, r)$ denotes the k -th order Reed-Muller code of length 2^r . In addition, the binary code VOA $M_{\text{RM}(2,4)}$ is isomorphic to $V_{\sqrt{2}E_8}^+$ (cf. [Mi04]). Similarly, $\mathcal{D}(d_{16}^+)^\perp = \text{span}_{\mathbb{Z}_2}\{d(\mathcal{E}_{16}), \ell(d_{16}^+)\}$ and the corresponding binary code VOA is isomorphic to $V_{\sqrt{2}D_{16}^+}^+$. Therefore, holomorphic

framed VOAs associated to subcodes of $\mathcal{D}(e_8)^{\oplus 3}$ and $\mathcal{D}(e_8) \oplus \mathcal{D}(d_{16}^+)$ are holomorphic extensions of $(V_{\sqrt{2}E_8}^+)^{\otimes 3}$ and $V_{\sqrt{2}E_8}^+ \otimes V_{\sqrt{2}D_{16}^+}^+$, respectively.

3. LATTICE TYPE VOA V_L^+ FOR A TOTALLY EVEN LATTICE L

Let L be an even lattice of rank $n \in 8\mathbb{Z}$. Assume that L is totally even, that is, $\sqrt{2}L^*$ is even, and that L has an orthogonal basis of norm 4. Then $L^* \subset L/2$. Let 2^m be the size of L^*/L . Since L has an orthogonal basis of norm 4, V_L^+ is a framed VOA [DMZ94]. By [DGH98], V_L^+ is rational and C_2 -cofinite. In this section, we review the properties of $V = V_L^+$ and the set $R(V)$ of all isomorphism classes of irreducible V -modules.

By [AD04], any irreducible V -module is isomorphic to one of the following:

$$\{V_{\lambda+L}^\pm, V_L^{T_{\chi\lambda}, \pm} \mid \lambda \in L^*/L\}.$$

Hence $|R(V)| = 2^{m+2}$. We refer to [Sh04] for the definition of χ_λ . We also use the following notations to denote the isomorphism classes:

$$[\lambda]^\pm = [V_{\lambda+L}^\pm], \quad [\chi_\lambda]^\pm = [V_L^{T_{\chi\lambda}, \pm}].$$

By [ADL05], the fusion rules of $R(V)$ are given as follows.

Proposition 3.1. (cf. [ADL05]) *Let L be an even lattice such that $\sqrt{2}L^*$ is even. Then the fusion rules of V_L^+ are described as follows:*

$$\begin{aligned} [\lambda_1]^\delta \boxtimes [\lambda_2]^\varepsilon &= [\lambda_1 + \lambda_2]^{\delta\varepsilon}, \\ [\lambda_1]^\delta \boxtimes [\chi_{\lambda_2}]^\varepsilon &= [\chi_{\lambda_1+\lambda_2}]^{\delta\varepsilon\nu(\lambda_2)\nu(\lambda_1+\lambda_2)}, \\ [\chi_{\lambda_1}]^\delta \boxtimes [\chi_{\lambda_2}]^\varepsilon &= [\lambda_1 + \lambda_2]^{\delta\varepsilon\nu(\lambda_1)\nu(\lambda_2)}, \end{aligned}$$

where $\nu(\lambda) = +$ and $-$ if $\langle \lambda, \lambda \rangle \in 2\mathbb{Z}$ and $1 + 2\mathbb{Z}$, respectively, and $\delta, \varepsilon \in \{\pm\} \cong \mathbb{Z}_2$.

By the proposition above, any irreducible V -module is a self-dual simple current module. By [ADL05], the associativity of \boxtimes holds for V_L^+ . Hence $R(V)$ has an elementary abelian 2-group structure of order 2^{m+2} under the fusion rules. We view $R(V)$ as an $(m+2)$ -dimensional vector space over \mathbb{F}_2 .

We now assume that $n \in 8\mathbb{Z}$. Then any irreducible V -module is graded by \mathbb{Z} or $\mathbb{Z} + 1/2$. Let q_V be the map from $R(V)$ to \mathbb{F}_2 defined by $q_V([M]) = 0$ and 1 if the weights of M belong to \mathbb{Z} and $1/2 + \mathbb{Z}$, respectively. Let \langle, \rangle be the 2-form on $R(V)$ defined by $\langle W, W' \rangle = q_V(W \boxtimes W') + q_V(W) + q_V(W')$. Then by [Sh04], it is a symplectic form, and hence q_V is a quadratic form. Moreover the type of q_V is equal to that of the quadratic form q_L on L^*/L defined by $q_L(v) = \langle v, v \rangle$. By Proposition 3.1, we obtain the following lemma directly.

Lemma 3.2. *Let $\lambda, \mu \in L^*/L$ and $\varepsilon, \delta \in \{\pm\}$. Then the following hold:*

- (1) $\langle [\lambda]^\varepsilon, [\mu]^\delta \rangle = 2\langle \lambda, \mu \rangle$;
- (2) $\langle [\lambda]^+, [\chi_0]^+ \rangle = 0$;
- (3) $\langle [\lambda]^-, [\chi_0]^+ \rangle = 1$;
- (4) $\langle [0]^-, [\chi_\lambda]^\varepsilon \rangle = 1$.

3.1. **VOA $V_{\sqrt{2}E_8}^+$.** Let E_8 denote the E_8 root lattice and set $\sqrt{2}E_8 = \{\sqrt{2}v \mid v \in E_8\}$. Then $\sqrt{2}E_8$ is totally even, and contains an orthogonal basis of norm 4. In this subsection, we review the properties of $V = V_{\sqrt{2}E_8}^+$.

By the previous section, $(R(V), q_V)$ is a 10-dimensional quadratic space of plus type over \mathbb{F}_2 . By the definition of q_V , $\text{Aut}(V)$ preserves it. Hence we obtain a group homomorphism from $\text{Aut}(V)$ to the orthogonal group $O(R(V), q_V)$. In fact, it is an isomorphism by [Sh04, Theorem 4.5] (cf. [Gr98]).

Let M be an irreducible module of $V = V_{\sqrt{2}E_8}^+$. Then the lowest weight of M is $0, 1/2$ or 1 , and it is 0 if and only if $M \cong V_{\sqrt{2}E_8}^+$ (cf. [AD04, Mi04]).

Since $V_{\sqrt{2}E_8} = V \oplus V_{\sqrt{2}E_8}^-$ is a VOA, the invariant bilinear form on the irreducible V -module $V_{\sqrt{2}E_8}^-$ is symmetric ([FHL93, Proposition 5.3.6]). By Lemma 1.2 (1), $\text{Aut}(V) \cong O(R(V), q_V)$ is transitive on the set of all isomorphism classes of \mathbb{Z} -graded irreducible V -modules except for $[0]^+$. Hence the invariant bilinear form on arbitrary \mathbb{Z} -graded irreducible V -module is also symmetric. Since the dimension of the lowest weight space of any $(\mathbb{Z} + 1/2)$ -graded irreducible V -module is one-dimensional, the invariant form on it must be symmetric.

Lemma 3.3. *Let $V = V_{\sqrt{2}E_8}^+$. Then the following hold:*

- (1) V is simple, rational, C_2 -cofinite, self-dual and of CFT type;
- (2) $(R(V), q_V)$ is a non-singular 10-dimensional quadratic space of plus type over \mathbb{F}_2 ;
- (3) $\text{Aut}(V) \cong O(R(V), q_V)$;
- (4) The invariant bilinear form on arbitrary irreducible V -module is symmetric;
- (5) For $[M] \in R(V)$, the lowest weight of M is $0, 1/2$ or 1 , and the dimension of the lowest weight space is $1, 1$ or 8 , respectively.

The following lemma will be used in the later sections.

Lemma 3.4. (cf. [Sh11]) *The automorphism group $\text{Aut}(V^k)$ of V^k is isomorphic to $\text{Aut}(V) \wr \text{Sym}_k$.*

3.2. **VOA $V_{\sqrt{2}D_{16}}^+$.** Let D_{16}^+ be an even unimodular lattice of rank 16 whose root lattice is D_{16} . Note that D_{16}^+ is a unique indecomposable even unimodular lattice of rank 16 up

to isomorphism. Set $\sqrt{2}D_{16}^+ = \{\sqrt{2}v \mid v \in D_{16}^+\}$. Then $\sqrt{2}D_{16}^+$ is totally even, and it contains an orthogonal basis of norm 4. In this subsection, we review the properties of $X = V_{\sqrt{2}D_{16}^+}^+$.

Let us recall a description of $\sqrt{2}D_{16}^+$. Let $\{\alpha_i \mid 1 \leq i \leq 16\}$ be an orthogonal basis of \mathbb{R}^{16} of norm 2 and let $\alpha_c = \sum_{i=1}^{16} c_i \alpha_i$ for $c = (c_i) \in \mathbb{F}_2^{16}$. Then

$$\sqrt{2}D_{16}^+ \cong \sum_{1 \leq i, j \leq 16} \mathbb{Z}(\alpha_i + \alpha_j) + \mathbb{Z} \frac{\alpha_{(1^{16})}}{2}.$$

It is easy to see that

$$(3.1) \quad (\sqrt{2}D_{16}^+)^* \cong \sum_{1 \leq i \leq 16} \mathbb{Z}\alpha_i + \sum_{c \in \mathcal{E}_{16}} \mathbb{Z} \frac{\alpha_c}{2} + \mathbb{Z} \frac{\alpha_{(1^{16})}}{4},$$

where \mathcal{E}_{16} is the binary code of length 16 consisting of all codewords with even weight.

Lemma 3.5. *Let $c \in \mathbb{F}_2^{16}$ with $\text{wt}(c) = 8$. Then $\sqrt{2}D_{16}^+ + \mathbb{Z}\alpha_c/2$ contains a sublattice isometric to $\sqrt{2}E_8^{\oplus 2}$.*

Proof. This is clear from

$$\sqrt{2}E_8 \cong \sum_{1 \leq i, j \leq 8} \mathbb{Z}(\alpha_i + \alpha_j) + \mathbb{Z} \frac{\alpha_{(1^8)}}{2}.$$

□

By the method for calculating the automorphism group of V_L^+ given in [Sh04], one can show that $\text{Aut}(X) \cong 2^{15} \cdot (2^{14} \cdot \text{Sym}_{16}) \cdot \text{Sym}_3$. The orbits can then be computed directly. By the explicit calculation, one can also show that the invariant form among $[\lambda]^\pm$ is symmetric (cf. [FLM88]). Hence the invariant form on any irreducible V -modules is symmetric since each orbit in Table 3 contains an element of the form $[\lambda]^\pm$.

The fusion rules of X and the quadratic form q_X on $R(X)$ was described in Section 3 (See Proposition 3.1 and Lemma 3.2). Hence we obtain the following lemma.

Lemma 3.6. *Let $X = V_{\sqrt{2}D_{16}^+}^+$. Then the following hold:*

- (1) X is simple, rational, C_2 -cofinite, self-dual and of CFT type;
- (2) $(R(X), q_X)$ is a non-singular 18-dimensional quadratic space of plus type over \mathbb{F}_2 ;
- (3) The invariant bilinear form on arbitrary irreducible X -module is symmetric;
- (4) The orbits in $R(X)$ for the action of $\text{Aut}(X)$ are given by Table 3.

TABLE 3. Orbits of irreducible modules of $V_{\sqrt{2}D_{16}}^+$ under $\text{Aut}(V_{\sqrt{2}D_{16}}^+)$.

orbits	orbit size	lowest weight	dim. of lowest space
$[0]^+$	1	0	1
$[0]^-, [\alpha_1]^\pm$,	3	1	16
$[\alpha_c/2]^\pm, [\alpha_c/2 - \alpha_1]^\pm$ $\text{wt}(c) = 2$,	$2^2 \times 120$	$1/2$	1
$[\alpha_c/2]^\pm, [\alpha_c/2 - \alpha_1]^\pm$ $\text{wt}(c) = 4$,	$2^2 \times 1820$	1	4
$[\alpha_c/2]^\pm, [\alpha_c/2 - \alpha_1]^\pm$ $\text{wt}(c) = 6$,	$2^2 \times 8008$	$3/2$	16
$[\alpha_c/2]^\pm, [\alpha_c/2 - \alpha_1]^\pm$ $\text{wt}(c) = 8$	2×12870	2	128
$[\alpha_{(1^{16})}/4 - \alpha_c/2]^\pm, (c \in \mathcal{E}_{16}), [\chi_\lambda]^+$	$2^{15} + 2^{16}$	1	1
$[\alpha_{(-31^{15})}/4 - \alpha_c/2]^\pm, (c \in \mathcal{E}_{16}), [\chi_\lambda]^-$	$2^{15} + 2^{16}$	$3/2$	16

4. FRAMED VOAs ASSOCIATED TO SUBCODES OF $\mathcal{D}(e_8)^{\oplus 3}$

In this section, we will study the framed VOAs of central charge 24 associated to subcodes of $\mathcal{D}(e_8)^{\oplus 3}$. Recall that $\mathcal{D}(e_8) \cong \text{RM}(1, 4)$ and $\text{RM}(1, 4)^\perp = \text{RM}(2, 4)$ and the binary code VOA $M_{\text{RM}(2, 4)}$ is isomorphic to $V_{\sqrt{2}E_8}^+$. Therefore, if U is a holomorphic framed VOA whose 1/16 code is contained in $\mathcal{D}(e_8)^{\oplus 3}$, then U contains $(V_{\sqrt{2}E_8}^+)^{\otimes 3}$ as a full subVOA and U is a holomorphic simple current extension of $(V_{\sqrt{2}E_8}^+)^{\otimes 3}$.

4.1. Simple current extensions of $(V_{\sqrt{2}E_8}^+)^{\otimes k}$. Let $V = V_{\sqrt{2}E_8}^+$. For the detail of V , see Section 3.1. In this subsection, we study holomorphic VOAs associated to maximal totally singular subspaces of $(R(V)^k, q_V^k)$, which correspond to holomorphic simple current extensions of V^k .

We will recall the relation between simple current extensions of $(V_{\sqrt{2}E_8}^+)^{\otimes k}$ and totally singular subspaces of $R(V)^k$ from [Sh11].

Let k be a positive integer. We identify $R(V^k)$ with $R(V)^k$ by Lemma 1.3. By Lemma 3.3 (2), $(R(V)^k, q_V^k)$ is a non-singular $10k$ -dimensional quadratic space of plus type over \mathbb{F}_2 .

Notation 4.1. Let \mathcal{T} be a subset of $R(V)^k$. We define $\mathfrak{V}(\mathcal{T}) = \oplus_{[M] \in \mathcal{T}} M$.

If \mathcal{T} is a totally singular subspace, then $\mathfrak{V}(\mathcal{T}) = \oplus_{[M] \in \mathcal{T}} M$ is a VOA, which is a simple current extension of V^k . Conversely, let U be a simple current extension of V^k . Then $U \cong \mathfrak{V}(\mathcal{T}) = \oplus_{[M] \in \mathcal{T}} M$ for some totally singular subspace \mathcal{T} of $R(V)^k$.

Proposition 4.2. [Sh11, Proposition 4.4] *Let $V = V_{\sqrt{2}E_8}^+$. Then the V^k -module $\mathfrak{V}(\mathcal{T}) = \oplus_{[M] \in \mathcal{T}} M$ has a simple VOA structure of central charge $8k$ by extending its V^k -module*

structure if and only if \mathcal{T} is a totally singular subspace of $R(V)^k$. Moreover, $\mathfrak{V}(\mathcal{T})$ is holomorphic if and only if \mathcal{T} is maximal.

Since V is framed, so is $\mathfrak{V}(\mathcal{T})$. By Lemmas 3.3 (3) and 3.4, we have $\text{Aut}(V^k) \cong O(R(V), q_V) \wr \text{Sym}_k$. By Lemma 1.4, we obtain the following lemma.

Lemma 4.3. *Let \mathcal{T} be a totally singular subspace of $R(V)^k$ and let g be an automorphism of V^k . Then the VOA $\mathfrak{V}(g \circ \mathcal{T})$ is isomorphic to $\mathfrak{V}(\mathcal{T})$.*

Lemma 4.4. *Let \mathcal{S} be a maximal totally singular subspace of $R(V)^k$.*

- (1) *If \mathcal{S} contains $(a_1, 0, \dots, 0), (0, a_2, 0, \dots, 0), \dots$, and $(0, \dots, 0, a_k)$ for some $a_i \in S(R(V))^\times$, $i = 1, \dots, k$, then $\mathfrak{V}(\mathcal{S})$ is isomorphic to a lattice VOA V_L .*
- (2) *If \mathcal{S} contains $(a_1, a_2, 0, \dots, 0), (0, a_2, a_3, 0, \dots, 0), \dots$, and $(0, \dots, 0, a_{k-1}, a_k)$ for some $a_i \in S(R(V))^\times$, $i = 1, \dots, k$, then $\mathfrak{V}(\mathcal{S})$ is isomorphic to V_L or its \mathbb{Z}_2 -orbifold \tilde{V}_L .*

Proof. Recall that a_i conjugate to $[0]^-$ by Lemma 3.3.

If \mathcal{S} satisfies the assumption of (1) then by Lemma 4.3, $\mathfrak{V}(\mathcal{S})$ contains a full subVOA isomorphic to $V_{\sqrt{2}E_8^{\oplus k}}$. Hence we may view $\mathfrak{V}(\mathcal{S})$ as a simple current extension of $V_{\sqrt{2}E_8^{\oplus k}}$, and $\mathfrak{V}(\mathcal{S})$ is isomorphic to V_L for some even overlattice L of $\sqrt{2}E_8^{\oplus k}$.

If \mathcal{S} satisfies the assumption of (2) then by Lemma 4.3, $\mathfrak{V}(\mathcal{S})$ contains a full subVOA isomorphic to $V_{\sqrt{2}E_8^{\oplus k}}^+$. Hence we may view $\mathfrak{V}(\mathcal{S})$ as a simple current extension of $V_{\sqrt{2}E_8^{\oplus k}}^+$, and $\mathfrak{V}(\mathcal{S})$ is isomorphic to V_L or \tilde{V}_L for some even overlattice L of $\sqrt{2}E_8^{\oplus k}$. \square

4.2. Construction of maximal totally singular subspaces of (R^3, q^3) . Let (R, q) be a non-singular $2m$ -dimensional quadratic space of plus type over \mathbb{F}_2 . Then (R^3, q^3) is a non-singular $6m$ -dimensional quadratic space of plus type over \mathbb{F}_2 . In this subsection, we study maximal totally singular subspaces \mathcal{S} of (R^3, q^3) .

Let us consider the following two conditions on maximal totally singular subspaces \mathcal{S} of R^3 :

$$(4.1) \quad (a, 0, 0), (0, b, 0), (0, 0, c) \in \mathcal{S} \setminus \{0\} \text{ for some } a, b, c \in S(R)^\times,$$

$$(4.2) \quad (a, b, 0), (0, b, c) \in \mathcal{S} \setminus \{0\} \text{ for some } a, b, c \in S(R)^\times,$$

where $S(R)^\times$ is the set of all non-zero singular vectors in R .

Remark 4.5. If $R = R(V_{\sqrt{2}E_8}^+)$, then by Lemma 4.4, the VOA $\mathfrak{V}(\mathcal{S})$ associated to a maximal totally singular subspace \mathcal{S} satisfying (4.1) or (4.2) is isomorphic to a lattice VOA V_L or its \mathbb{Z}_2 -orbifold \tilde{V}_L .

Now let us construct maximal totally singular subspaces satisfying neither (4.1) nor (4.2).

Theorem 4.6. *Let S_1 be a k_1 -dimensional totally singular subspace of R and let S_2 be a k_2 -dimensional totally singular subspace of S_1 . Assume that $m - k_1 - k_2$ is even.*

Let P be an $(m - k_1 - k_2)$ -dimensional non-singular subspace of S_1^\perp of ε type, where $\varepsilon \in \{\pm\}$. Let Q be a complementary subspace of S_1 in $(S_1 \perp P)^\perp$. Let T be a complementary subspace of S_2 in $(S_2 \perp P)^\perp$. Let $U = Q^\perp$. Then the following hold:

- (1) *T and U are non-singular isomorphic quadratic spaces;*
- (2) *Let φ be an isomorphism of quadratic spaces from T to U and let $\mathcal{S}(S_1, S_2, P, Q, T, \varphi)$ be the subspace of R^3 defined by*

$$\text{Span}_{\mathbb{F}_2} \left\{ (s_1, 0, 0), (0, s_2, 0), (p, p, 0), (q, 0, q), (0, t, \varphi(t)) \mid s_i \in S_i, p \in P, q \in Q, t \in T \right\}.$$

Then $\mathcal{S}(S_1, S_2, P, Q, T, \varphi)$ is a maximal totally singular subspace of R^3 ;

- (3) *$\mathcal{S}(S_1, S_2, P, Q, T, \varphi)$ depends only on k_1, k_2 and ε up to $O(R, q) \wr \text{Sym}_3$;*
- (4) *$\mathcal{S}(S_1, S_2, P, Q, T, \varphi)$ satisfies neither (4.1) nor (4.2).*

Proof. It is easy to see that

$$(4.3) \quad \dim Q = m - k_1 + k_2 \quad \text{and} \quad \dim T = \dim U = m + k_1 - k_2.$$

Since S_i is the radical of S_i^\perp , both T and Q are non-singular, and so is U . Since $P \perp Q$, $P \perp T$ and $Q \perp U$ are of $+$ type, the types of P, Q, T, U are ε , and we obtain (1).

Clearly the generators of $\mathcal{S}(S_1, S_2, P, Q, T, \varphi)$ are singular and they are perpendicular to each other. Hence $\mathcal{S}(S_1, S_2, P, Q, T, \varphi)$ is totally singular. Since

$$\dim \mathcal{S}(S_1, S_2, P, Q, T, \varphi) = \dim S_1 + \dim S_2 + \dim P + \dim Q + \dim T = 3m,$$

it is maximal totally singular. Hence we obtain (2).

Consider $\mathcal{S}(S'_1, S'_2, P', Q', T', \varphi')$ under the assumption that $\dim S'_i = \dim S_i = k_i$ for $i = 1, 2$ and the type of P' is ε . Up to the actions of $O(R, q)$ on the first and second coordinates, we may assume that $S'_1 = S_1$ and $S'_2 = S_2$. Moreover, by the action of $\text{Stab}_{O(R, q)}(S_i)$ on each coordinate, we may assume that $P' = P$, $Q' = Q$ and that $(p, p, 0), (q, 0, q) \in \mathcal{S}(S'_1, S'_2, P', Q', T', \varphi')$ for all $p \in P$ and $q \in Q$ (cf. Lemma 1.2 (2)). Up to the action of $\text{Stab}_{O(R, q)}(S_2) \cap \text{Stab}_{O(R, q)}^{\text{pt}}(Q)$ on the second coordinate, we obtain $T' = T$. Furthermore, by the action of $\text{Stab}_{O(R, q)}^{\text{pt}}(Q)$ on the third coordinate, we may assume that $\varphi' = \varphi$, and hence we obtain (3).

(4) follows from the definition of $\mathcal{S}(S_1, S_2, P, Q, T, \varphi)$. □

Remark 4.7. By (3), we denote $\mathcal{S}(S_1, S_2, P, Q, T, \varphi)$ by $\mathcal{S}(m, k_1, k_2, \varepsilon)$.

Theorem 4.8. *Let S_1 be a k_1 -dimensional totally singular subspace of R and let S_2 be a k_2 -dimensional totally singular subspace of S_1 . Assume that $m - k_1 - k_2$ is odd.*

Let P be an $(m - k_1 - k_2 - 1)$ -dimensional non-singular subspace of S_1^\perp of plus type. Let Q be an $(m - k_1 + k_2 - 1)$ -dimensional non-singular subspace of $(S_1 \perp P)^\perp$ of plus type. Let B be a complementary subspace of S_1 in $(S_1 \perp P \perp Q)^\perp$. Let T be a complementary subspace of S_2 in $(P \perp S_2 \perp B)^\perp$. Let $U = (Q \perp B)^\perp$. Then the following hold:

- (1) *B is a 2-dimensional non-singular subspace of plus type;*
- (2) *T and U are isomorphic non-singular quadratic spaces of plus type;*
- (3) *Let y be the non-singular vector in B and let z be a non-zero singular vector in B . Let φ be an isomorphism of quadratic spaces from T to U and set*

$$\mathcal{S}(S_1, S_2, P, Q, B, T, z, \varphi) = \text{Span}_{\mathbb{F}_2} \left\{ (s_1, 0, 0), (0, s_2, 0), (p, p, 0), (q, 0, q), (0, t, \varphi(t)), (y, y, 0), (y, 0, y), (z, z, z) \right. \\ \left. \mid s_i \in S_i, p \in P, q \in Q, t \in T \right\}.$$

Then $\mathcal{S}(S_1, S_2, P, Q, B, T, z, \varphi)$ is a maximal totally singular subspace of R^3 ;

- (4) *$\mathcal{S}(S_1, S_2, P, Q, B, T, z, \varphi)$ depends only on k_1, k_2 up to $O(R, q) \wr \text{Sym}_3$;*
- (5) *$\mathcal{S}(S_1, S_2, P, Q, B, T, z, \varphi)$ satisfies neither (4.1) nor (4.2).*

Proof. It is easy to see that $\dim B = (2m - 2k_1) - (m - k_1 - k_2 - 1) - (m - k_1 + k_2 - 1) = 2$. Since S_1 is the radical of S_1^\perp , B is non-singular. Since P and Q are of plus type, so is B .

It is easy to see that

$$(4.4) \quad \dim T = \dim U = m + k_1 - k_2 - 1.$$

Since S_2 is the radical of S_2^\perp , T is non-singular. Since the type of P and Q are the same, we obtain (2).

Clearly the generators of $\mathcal{S}(S_1, S_2, P, Q, B, T, z, \varphi)$ are singular and they are perpendicular to each other. Hence $\mathcal{S}(S_1, S_2, P, Q, B, T, z, \varphi)$ is totally singular. Since

$$\dim \mathcal{S}(S_1, S_2, P, Q, B, T, z, \varphi) = \dim S_1 + \dim S_2 + \dim P + \dim Q + \dim T + 3 = 3m,$$

it is maximal totally singular. Hence we obtain (3).

Consider $\mathcal{S}(S'_1, S'_2, P', Q', B', T', z', \varphi')$ under the assumption that $\dim S'_i = \dim S_i = k_i$ for $i = 1, 2$ and the type of P' is plus. Up to the actions of $O(R, q)$ on the first and second coordinates, we may assume that $S'_1 = S_1$ and $S'_2 = S_2$. Moreover, up to the action of $\text{Stab}_{O(R, q)}(S_i)$ on each coordinate (cf. Lemma 1.2 (2)), we may assume that $P' = P$, $Q' = Q$, $B' = B$, $z' = z$ and that $(p, p, 0), (q, 0, q) \in \mathcal{S}(S'_1, S'_2, P', Q', B', T', z', \varphi')$

for all $p \in P$ and $q \in Q$. Up to the action of $\text{Stab}_{O(R,q)}(S_2) \cap \text{Stab}_{O(R,q)}^{\text{pt}}(Q \perp B)$ on the second coordinate, we may assume that $T' = T$. Furthermore, up to the action of $\text{Stab}_{O(R,q)}^{\text{pt}}(Q \perp B)$ in $O(R, q)$ on the third coordinate, we may assume that $\varphi' = \varphi$, and hence we obtain (4).

(5) follows from the definition of $\mathcal{S}(S_1, S_2, P, Q, B, T, z, \varphi)$. \square

Remark 4.9. By (4), we often denote $\mathcal{S}(S_1, S_2, P, Q, B, T, z, \varphi)$ by $\mathcal{S}(m, k_1, k_2)$.

Next we count the numbers of certain vectors in $\mathcal{S}(m, k_1, k_2, \varepsilon)$ and $\mathcal{S}(m, k_1, k_2)$.

Lemma 4.10. *Let $\mathcal{S} = \mathcal{S}(m, k_1, k_2, \varepsilon)$ or $\mathcal{S}(m, k_1, k_2)$.*

(1) *The number of vectors in \mathcal{S} of the form $\sigma(a, 0, 0)$, $a \in S(R)^\times$ and $\sigma \in \text{Sym}_3$, is $2^{k_1} + 2^{k_2} - 2$.*

(2) *The number of vectors in \mathcal{S} of the form $\sigma(b, c, 0)$, $b, c \in \overline{S(R)}$ and $\sigma \in \text{Sym}_3$, is*

$$\begin{cases} 2^{m-1} + 2^{m+k_1-1} + 2^{m+k_2-1} - 3 \times 2^{(m+k_1+k_2)/2-1} & \text{if } \varepsilon = +, m - k_1 - k_2 \in 2\mathbb{Z}, \\ 2^{m-1} + 2^{m+k_1-1} + 2^{m+k_2-1} + 3 \times 2^{(m+k_1+k_2)/2-1} & \text{if } \varepsilon = -, m - k_1 - k_2 \in 2\mathbb{Z}, \\ 2^{m-1} + 2^{m+k_2-1} + 2^{m+k_1-1} & \text{if } m - k_1 - k_2 \in 1 + 2\mathbb{Z}. \end{cases}$$

Proof. The number of vectors in (1) is equal to the number of all nonzero vectors in S_1 and S_2 , which is $2^{k_1} + 2^{k_2} - 2$.

Let $v = (a_1, a_2, a_3)$ be a vector in (2). Then one of a_i is zero. Assume that $m - k_1 - k_2 \in 2\mathbb{Z}$. If $a_3 = 0$ then $a_1 \in a_2 + S_1$, $a_2 \in \overline{S(P)} + S_2$. The number of such vectors is $2^{k_1+k_2} |\overline{S(P)}|$. If $a_2 = 0$ then $a_1 \in a_3 + S_1$, $a_3 \in \overline{S(Q)}$. The number of such vectors is $2^{k_1} |\overline{S(Q)}|$. If $a_1 = 0$ then $a_2 \in a_3 + S_2$, $a_3 \in \overline{S(U)}$. The number of such vectors is $2^{k_2} |\overline{S(U)}|$.

Assume that $m - k_1 - k_2 \in 2\mathbb{Z} + 1$. If $a_3 = 0$ then $a_1 \in a_2 + S_1$ and, $a_2 \in \overline{S(P)} + S_2$ or $a_2 \in y + S(P) + S_2$. The number of such vectors is $2^{k_1+k_2} |P|$. If $a_2 = 0$ then $a_1 \in a_3 + S_1$ and, $a_3 \in \overline{S(Q)}$ or $a_3 \in y + S(Q)$. The number of such vectors is $2^{k_1} |Q|$. If $a_1 = 0$ then $a_2 \in a_3 + S_2$ and, $a_3 \in \overline{S(U)}$ or $a_3 \in y + S(U)$. The number of such vectors is $2^{k_2} |U|$. \square

4.3. Classification of maximal totally singular subspaces of (R^3, q^3) . Let ρ_i denote the i -th coordinate projection $R^3 \rightarrow R$, $(a_1, a_2, a_3) \mapsto a_i$. For a subspace \mathcal{S} of R^3 and distinct $i, j \in \{1, 2, 3\}$, we denote $\mathcal{S}^{(i)} = \{v \in \mathcal{S} \mid \rho_i(v) = 0\}$ and $\mathcal{S}^{(ij)} = \{v \in \mathcal{S} \mid \rho_i(v) = \rho_j(v) = 0\}$. The next theorem classifies all maximal totally singular subspaces of R^3 , up to $O(R, q) \wr \text{Sym}_3$.

Theorem 4.11. *Let \mathcal{S} be a maximal totally singular subspace of R^3 . Then up to $O(R, q) \wr \text{Sym}_3$, one of the following holds:*

(1) \mathcal{S} satisfies (4.1);

(2) \mathcal{S} satisfies (4.2);

(3) \mathcal{S} is conjugate to $\mathcal{S}(S_1, S_2, P, Q, T, \varphi)$ defined as in Theorem 4.6;

(4) \mathcal{S} is conjugate to $\mathcal{S}(S_1, S_2, P, Q, B, T, z, \varphi)$ defined as in Theorem 4.8.

Proof. Let \mathcal{S} be a maximal totally singular subspace of R^3 satisfying neither (4.1) nor (4.2). Set $S_1 = \rho_1(\mathcal{S}^{(23)})$, $S_2 = \rho_2(\mathcal{S}^{(13)})$, $S_3 = \rho_3(\mathcal{S}^{(12)})$ and $k_i = \dim S_i$. Up to the action of $\text{Sym}_3 \subset O(R, q) \wr \text{Sym}_3$, we may assume that $k_1 \geq k_2 \geq k_3$. If $k_3 \geq 1$ then \mathcal{S} satisfies (4.1), which is a contradiction. Hence $k_3 = 0$, and $\mathcal{S}^{(12)} = 0$. Up to the action of $O(R, q)$ on the second coordinate, we may assume that $S_2 \subset S_1$.

Let \mathcal{P} , \mathcal{Q} , \mathcal{T} be complementary subspaces of $\mathcal{S}^{(23)} \perp \mathcal{S}^{(13)}$ in $\mathcal{S}^{(3)}$, $\mathcal{S}^{(23)}$ in $\mathcal{S}^{(2)}$, and $\mathcal{S}^{(13)}$ in $\mathcal{S}^{(1)}$, respectively. By the maximality of \mathcal{S} , we have $\rho_i(\mathcal{S}) = S_i^\perp$, and hence $\dim \rho_3(\mathcal{S}) = 2m$, $\dim \rho_2(\mathcal{S}) = 2m - k_2$ and $\dim \rho_1(\mathcal{S}) = 2m - k_1$. It is easy to see that

$$\dim \mathcal{P} = m - k_1 - k_2, \quad \dim \mathcal{Q} = m - k_1 + k_2, \quad \dim \mathcal{T} = m + k_1 - k_2.$$

Since \mathcal{S} is totally singular, $\langle \rho_i(\mathcal{P}), \rho_i(\mathcal{Q}) \rangle = \langle \rho_i(\mathcal{P}), \rho_i(\mathcal{T}) \rangle = \langle \rho_i(\mathcal{T}), \rho_i(\mathcal{Q}) \rangle = 0$ for $i = 1, 2, 3$. By the dimensions,

$$(4.5) \quad (\rho_1(\mathcal{P}) \perp S_1)^\perp = \rho_1(\mathcal{Q}) \perp S_1,$$

$$(4.6) \quad (\rho_2(\mathcal{P}) \perp S_2)^\perp = \rho_2(\mathcal{T}) \perp S_2,$$

$$(4.7) \quad (\rho_3(\mathcal{Q}))^\perp = \rho_3(\mathcal{T})$$

in R .

Suppose that the dimension of the radical $\text{Rad}(\rho_1(\mathcal{P}))$ of $\rho_1(\mathcal{P})$ is greater than or equal to 2. Then by Lemma 1.2 (3), there exist non-zero singular vectors $a \in \text{Rad}(\rho_1(\mathcal{P}))$ and $b \in R$ such that $(a, b, 0) \in \mathcal{P}$. By (4.5), we have $a \in \rho_1(\mathcal{Q}) \perp S_1$. By the definition of \mathcal{P} , $a \notin S_1$. This shows that $\mathcal{Q} \perp \mathcal{S}^{(23)}$ contains $(a, 0, c)$ for some non-zero singular vector $c \in R$, which contradicts that \mathcal{S} does not satisfy (4.2). Thus $\dim \text{Rad}(\rho_1(\mathcal{P})) \leq 1$. Moreover, if $\dim \text{Rad}(\rho_1(\mathcal{P})) = 1$ then the non-zero vector must be non-singular. By the same arguments, we have $\dim \text{Rad}(\rho_1(\mathcal{Q})) \leq 1$ and $\dim \text{Rad}(\rho_2(\mathcal{T})) \leq 1$.

Assume that $\dim \text{Rad}(\rho_1(\mathcal{P})) = 0$. Then $m - k_1 - k_2$ is even. Let ε be the type of $\rho_1(\mathcal{P})$. Up to the action of $\text{Stab}_{O(R, q)}(S_2)$ on the second coordinate, we may assume that $\mathcal{P} = \{(p, p, 0) \mid p \in \rho_1(\mathcal{P})\}$. Moreover, up to the action of $O(R, q)$ on the third coordinate, we may assume that $\mathcal{Q} = \{(q, 0, q) \mid q \in \rho_1(\mathcal{Q})\}$. By (4.7), we have $\rho_3(\mathcal{T}) = (\rho_1(\mathcal{Q}))^\perp$. Let φ be the map from $\rho_2(\mathcal{T})$ to $\rho_3(\mathcal{T})$ defined by $(0, t, \varphi(t)) \in \mathcal{T}$. Since \mathcal{T} is a subspace and $q(t) = q(\varphi(t))$ for all $t \in \rho_2(\mathcal{T})$, φ is an isomorphism from $\rho_2(\mathcal{T})$ to $\rho_3(\mathcal{T})$. Thus \mathcal{S} is conjugate to $\mathcal{S}(S_1, S_2, \rho_1(\mathcal{P}), \rho_1(\mathcal{Q}), \rho_2(\mathcal{T}), \varphi)$ under $O(R, q) \wr \text{Sym}_3$.

Assume that $\dim \text{Rad}(\rho_1(\mathcal{P})) = 1$. Then $m - k_1 - k_2$ is odd. Let $\text{Rad}(\rho_1(\mathcal{P})) = \mathbb{F}_2 y$. By the argument in the third paragraph, y must be non-singular. Let P be a complementary subspace of $\mathbb{F}_2 y$ in $\rho_1(\mathcal{P})$. Since y is non-singular and is orthogonal to P , we may assume that P is of $+$ type. Up to the action of $\text{Stab}_{O(R,q)}(S_2)$ on the second coordinate, we may assume that $\mathcal{P} = \{(p, p, 0) \mid p \in P \perp \mathbb{F}_2 y\}$. By (4.5), $y \in \rho_1(\mathcal{Q}) \perp S_1$. Hence we may assume that $y \in \text{Rad}(\rho_1(\mathcal{Q}))$. Let Q be a complementary subspace of $\mathbb{F}_2 y$ in $\rho_1(\mathcal{Q})$. We may assume that Q is of $+$ type. Up to the action of $O(R, q)$ on the third coordinate, we may also assume that $\mathcal{Q} = \{(q, 0, q) \mid q \in Q \perp \mathbb{F}_2 y\}$. Since \mathcal{S} contains $(y, y, 0)$ and $(y, 0, y)$, we have $(0, y, y) \in \mathcal{T} \perp \mathcal{S}^{(13)}$. By (4.5), $y \in \rho_2(\mathcal{T}) \perp S_2$. Hence we may assume that $y \in \rho_2(\mathcal{T})$, and $(0, y, y) \in \mathcal{T}$. Let B be a complementary subspace of S_1 in $(S_1 \perp P \perp Q)^\perp$ such that $y \in B$. Let $T = B^\perp \cap \rho_2(\mathcal{T})$ and $U = B^\perp \cap \rho_3(\mathcal{T})$. Let φ be the map from T to U defined by $(0, t, \varphi(t)) \in \mathcal{T}$. Since \mathcal{T} is a subspace and $q(t) = q(\varphi(t))$ for all $t \in T$, φ is an isomorphism from T to U . Since both types of P and Q are $+$, the type of B is also $+$. Let z be a non-zero singular vector in B . Set $\mathcal{S}' = \text{Span}_{\mathbb{F}_2} \{\mathcal{S}^{(23)}, \mathcal{S}^{(13)}, \mathcal{P}, \mathcal{Q}, \mathcal{T}\}$. Then

$$(\mathcal{S}')^\perp / \mathcal{S}' = \text{Span}_{\mathbb{F}_2} \{(z, z, z) + \mathcal{S}', (y, 0, 0) + \mathcal{S}'\}.$$

Since y is non-singular, $(y, 0, 0) + \mathcal{S}' \notin \mathcal{S}/\mathcal{S}'$. Since \mathcal{S} is maximal totally singular, (z, z, z) or $(z + y, z, z) \in \mathcal{S}$. Up to the action of $\text{Stab}_{O(R,q)}(S_1) \cap \text{Stab}_{O(R,q)}^{\text{pt}}(P \perp Q \perp \mathbb{F}_2 y)$ on the first coordinate, we may assume that $(z, z, z) \in \mathcal{S}$. Therefore \mathcal{S} is conjugate to $\mathcal{S}(S_1, S_2, P, Q, B, T, z, \varphi)$ under $O(R, q) \wr \text{Sym}_3$. \square

4.4. Lie algebras of simple current extensions of $(V_{\sqrt{2}E_8}^+)^{\otimes k}$. In this subsection, we study the Lie algebra structure of $\mathfrak{W}(\mathcal{T})$ for a totally singular subspace \mathcal{T} of $R(V)^k$.

Let $S(R(V))$ denote the set of all singular vectors in $R(V)$. Set $S(R(V))^\times = S(R(V)) \setminus \{0\}$ and $\overline{S(R(V))} = R(V) \setminus S(R(V))$. The following lemma is easy.

Lemma 4.12. *Let $[M]$ be a vector in $(R(V)^k, q_V^k)$. Then $M_1 \neq 0$ if and only if $[M] = \sigma(a, 0, \dots, 0)$ or $[M] = \sigma(b, c, 0, \dots, 0)$ for some $a \in S(R(V))^\times$ and $b, c \in \overline{S(R(V))}$ and $\sigma \in \text{Sym}_k$.*

Define $d : R(V)^k \rightarrow \{0, 1, 8\} \subset \mathbb{Z}$ by

$$d(v) = \begin{cases} 8 & \text{if } v = \sigma(a, 0, \dots, 0) \text{ for some } a \in S(R(V))^\times, \sigma \in \text{Sym}_k, \\ 1 & \text{if } v = \sigma(b, c, 0, \dots, 0) \text{ for some } b, c \in \overline{S(R(V))}, \sigma \in \text{Sym}_k, \\ 0 & \text{otherwise.} \end{cases}$$

For a subset \mathcal{T} of $R(V)^k$, we set $d(\mathcal{T}) = \sum_{v \in \mathcal{T}} d(v)$.

Lemma 4.13. *Let \mathcal{T} be a totally singular subspace of $R(V)^k$. Then the dimension of the weight 1 subspace of $\mathfrak{V}(\mathcal{T})$ is equal to $d(\mathcal{T})$.*

Proof. This follows from Lemma 3.3 (5) and the definition of d . \square

Lemma 4.14. *Let M_a, M_b and M_c be irreducible modules for V^k . Assume that $(M_x)_1 \neq 0$ for $x = a, b, c$. Let $\mathcal{Y}(\cdot, z) = \sum_{i \in \mathbb{C}} a_{(i)} z^{i-1}$ be a non-zero intertwining operator of type $M_a \times M_b \rightarrow M_c$. Then for some non-zero vector $v \in (M_a)_1$, the map $v_{(0)} : (M_b)_1 \rightarrow (M_c)_1$ is non-zero.*

Proof. By the assumption, $(M_x)_1 \neq 0$ for $x \in \{a, b, c\}$ and $[M_a] \times [M_b] = [M_c]$. Note that the fusion rules are preserved by the conjugation action of $\text{Aut}(V^k)$ and thus, up to $\text{Aut}(V) \wr \text{Sym}_k$, we may assume that $[M_a] = ([0]^-, [0]^+, \dots, [0]^+)$ or $([\lambda]^+, [\lambda]^+[0]^+, \dots, [0]^+)$ by Lemma 4.12 (2), where λ is a vector in $(\sqrt{2}E_8)^*$ with norm 1.

By the explicit description of the intertwining operator ([FLM88]) for $[0]^-$ and $[\lambda]^+$, we have $v_{(0)} : (M_b)_1 \rightarrow (M_c)_1$ is non-zero for some $v \in (M_a)_1$. \square

Lemma 4.15. *Let \mathcal{T} be a totally singular subspace of $R(V)^k$ and \mathcal{U} a subspace of \mathcal{T} .*

- (1) *The subspace $\mathfrak{V}(\mathcal{U})_1$ is a Lie subalgebra of $\mathfrak{V}(\mathcal{T})_1$.*
- (2) *Assume that $(M^1 \boxtimes M^2)_1 = 0$ for all $[M^1] \in \mathcal{U}$, $[M^2] \in \mathcal{T} \setminus \mathcal{U}$ with $M_1^1 \neq 0$ and $M_1^2 \neq 0$. Then $\mathfrak{V}(\mathcal{U})_1$ is an ideal of $\mathfrak{V}(\mathcal{T})_1$.*
- (3) *Assume that $(M^1 \boxtimes M^2)_1 = 0$ for all $[M^1], [M^2] \in \mathcal{U}$ with $M_1^1 \neq 0$ and $M_1^2 \neq 0$. Then $\mathfrak{V}(\mathcal{U})_1$ is an abelian subalgebra.*

Proof. Since \mathcal{U} is a subspace, $\mathfrak{V}(\mathcal{U})$ is a subVOA of $\mathfrak{V}(\mathcal{T})$. Hence (1) holds.

For (2), let $[M^1] \in \mathcal{U}$ and $[M^2] \in \mathcal{T}$ with $M_1^1 \neq 0$ and $M_1^2 \neq 0$. By the assumption for (2), $((M^1)_1)_{(0)}(M^2)_1 \subset (M^3)_1 = 0$ if $[M^2] \in \mathcal{T} \setminus \mathcal{U}$, where $[M^3] = [M^1] \times [M^2]$. Hence (2) holds.

(3) can be proved by the similar arguments in (2). Note also that $[M] \times [M] = [V^k]$ for any $[M] \in \mathcal{T}$. Therefore, $(M_1)_{(0)}M_1 = 0$ since $V_1^k = 0$. \square

Lemma 4.16. *Let \mathcal{T} be a totally singular subspace of $R(V)^k$ and $[M]$ an element in \mathcal{T} with $M_1 \neq 0$. Then for any $a \in M_1$, $a_{(0)}$ is semisimple on $\mathfrak{V}(\mathcal{T})_1$.*

Proof. If $d([M]) = 8$ then by Lemma 1.2 (1), we may assume that $[M] = ([0]^-, [0]^+, \dots, [0]^+)$ up to $\text{Aut}(V^k)$. Since $(V_L^-)_1$ is the weight 1 subspace of V_L , one can see that its action on $R(V^k)$ is semisimple by [FLM88].

If $d([M]) = 1$ then $\dim M_1 = 1$. Hence the semisimplicity follows from the simple current property of M and Lemma 4.14. \square

4.5. **Lie algebras of holomorphic simple current extensions of $(V_{\sqrt{2}E_8}^+)^{\otimes 3}$.** In this subsection, we consider the case $k = 3$, and study the Lie algebra structure of the weight 1 subspace of $\mathfrak{V}(\mathcal{S})$ for a maximal totally singular subspace \mathcal{S} of $R(V)^3$. Recall that maximal totally singular subspaces of $(R(V)^3, q_V^3)$ were constructed in Theorems 4.6 and 4.8, and were classified in Theorem 4.11 up to $\text{Aut}(V^3) \cong \text{Aut}(V) \wr \text{Sym}_3$.

Combining Lemmas 4.10 and 4.13, we obtain the following.

Proposition 4.17. *Let $\mathcal{S} = \mathcal{S}(5, k_1, k_2, \varepsilon)$ or $\mathcal{S}(5, k_1, k_2)$ be the maximal totally singular subspace of $R(V)^3$ given in Theorem 4.6 or 4.8. Then the dimension of the weight 1 subspace of $\mathfrak{V}(\mathcal{S})$ is equal to*

$$d(\mathcal{S}) = \begin{cases} 3(2^{k_1+3} + 2^{k_2+3} - 2^{(3+k_1+k_2)/2}) & \text{if } \varepsilon = +, k_1 + k_2 \in 2\mathbb{Z} + 1, \\ 3(2^{k_1+3} + 2^{k_2+3} + 2^{(3+k_1+k_2)/2}) & \text{if } \varepsilon = -, k_1 + k_2 \in 2\mathbb{Z} + 1, \\ 3(2^{k_1+3} + 2^{k_2+3}) & \text{if } k_1 + k_2 \in 2\mathbb{Z}. \end{cases}$$

By Theorem 4.11 and the proposition above, we obtain the following corollary.

Corollary 4.18. *There are exactly 15 maximal totally singular subspaces of $R(V)^3$ satisfying neither (4.1) nor (4.2), up to $\text{Aut}(V^3)$. Moreover,*

$$\begin{aligned} d(\mathcal{S}(5, 1, 0, +)) &= 60, & d(\mathcal{S}(5, 1, 0, -)) &= 84, & d(\mathcal{S}(5, 3, 0, +)) &= 192, \\ d(\mathcal{S}(5, 3, 0, -)) &= 240, & d(\mathcal{S}(5, 5, 0, +)) &= 744, & d(\mathcal{S}(5, 2, 1, +)) &= 120, \\ d(\mathcal{S}(5, 2, 1, -)) &= 168, & d(\mathcal{S}(5, 4, 1, +)) &= 384, & d(\mathcal{S}(5, 3, 2, +)) &= 240, \\ d(\mathcal{S}(5, 0, 0)) &= 48, & d(\mathcal{S}(5, 2, 0)) &= 120, & d(\mathcal{S}(5, 4, 0)) &= 408, \\ d(\mathcal{S}(5, 1, 1)) &= 96, & d(\mathcal{S}(5, 3, 1)) &= 240, & d(\mathcal{S}(5, 2, 2)) &= 192. \end{aligned}$$

Let us study the Lie algebra structure of $\mathfrak{V}(\mathcal{S})$ for \mathcal{S} given in Theorems 4.6 and 4.8 by using their explicit descriptions.

Lemma 4.19. *Let $\mathcal{S} = \mathcal{S}(5, k_1, k_2, \varepsilon)$ or $\mathcal{S}(5, k_1, k_2)$.*

- (1) *If $k_1 \geq 1$ then the rank of $\mathfrak{V}(\mathcal{S})_1$ is greater than or equal to 8.*
- (2) *If $k_1 \geq k_2 \geq 1$ then the rank of $\mathfrak{V}(\mathcal{S})_1$ is equal to 16.*

Proof. Let $a \in S_1 \setminus \{0\}$. Then $\mathfrak{V}(\{(a, 0, 0)\})_1$ is an 8-dimensional abelian subalgebra of $\mathfrak{V}(\mathcal{S})_1$ since a is conjugate to $[0]^-$. Moreover, by Lemma 4.16, it is toral. Hence (1) follows.

Assume that $k_1 \geq k_2 \geq 1$. Let $s_i \in S_i$ for $i = 1, 2$. Set $H = \mathfrak{V}(\{(s_1, 0, 0), (0, s_2, 0)\})_1$. Then by Lemmas 4.12 and 4.16, H is toral abelian and $\dim H = 16$. Let us show that H is maximal abelian.

Let $v \in \mathfrak{V}(\mathcal{S})_1$ such that $v_{(0)}H = 0$. It suffices to show that $v \in H$. Let $v = \sum_{[M] \in \mathcal{S}} v_M$, where $v_M \in M_1$. Take $[M] = (a, b, c) \in \mathcal{S}$ with $v_M \neq 0$. Clearly, $M_1 \neq 0$. By Lemma 4.12, $[M] = (a, 0, 0)$, $(0, b, 0)$ for some $a, b \in S(R(V))^\times$ or $[M] = (a, b, 0)$, $(a, 0, c)$, $(0, b, c)$ for some $a, b, c \in \overline{S(R(V))}$.

First, we consider the case where $a, b \in \overline{S(R(V))}$ and $c = 0$. Then by the definitions of $\mathcal{S}(5, k_1, k_2, \varepsilon)$ and $\mathcal{S}(5, k_1, k_2)$, $a, b \in S_1^\perp$ and hence $a + s_1 \in \overline{S(R(V))}$. Since \mathcal{S} is a subspace, $(a + s_1, b, 0) = (a, b, 0) + (s_1, 0, 0) \in \mathcal{S}$. Let $[M'] = (a + s_1, b, 0)$. Then $M'_1 \neq 0$ and by Lemma 4.14, there is $h \in \mathfrak{V}(\{(s_1, 0, 0)\})_1 \subset H$ such that $(v_M)_{(0)}h \neq 0$. Since the projection of $v_{(0)}h$ to M' is $(v_M)_{(0)}h$, we have $v_{(0)}h \neq 0$, which contradicts $v_{(0)}H = 0$. Hence $v_M = 0$. By the same arguments, $v_M = 0$ if $[M] = (a, b, 0)$, $(a, 0, c)$ or $(0, b, c)$ with $a, b, c \in \overline{S(R(V))}$.

Next, we consider the case where $[M] = (a, 0, 0)$, $(0, b, 0)$ for some $a, b \in S(R(V))^\times$. Then $[M]$ belongs to $\{(x, y, 0) \mid x \in S_1, y \in S_2\}$. Since $v_{(0)}H = 0$, it is easy to see that $M_1 \subset H$ by the similar arguments. Hence $v_M \in H$, and $v \in H$. Thus H is maximal abelian, and the rank of $\mathfrak{V}(\mathcal{S})_1$ is 16. \square

Lemma 4.20. *Let $\mathcal{S} = \mathcal{S}(5, k_1, k_2, \varepsilon)$ or $\mathcal{S}(5, k_1, k_2)$.*

- (1) *If $k_1 = 2, 3$ and 4 then $\mathfrak{V}(\mathcal{S})_1$ has a semi-simple Lie subalgebra of type $(A_{1,1})^8$, $(D_{4,1})^2$ and $D_{8,1}$, respectively.*
- (2) *If $k_1 \geq k_2 \geq 2$ then $\mathfrak{V}(\mathcal{S})_1$ has a semi-simple Lie subalgebra of type $(A_{1,1})^{16}$.*
- (3) *If $k_1 = 3$ and $k_2 = 2$ then $\mathfrak{V}(\mathcal{S})_1$ has a semi-simple Lie subalgebra of type $(D_{4,1})^2(A_{1,1})^8$.*

Proof. Recall that for 2, 3 and 4 -dimensional totally singular subspaces U of $R(V)$, $\oplus_{[M] \in U} M$ are lattice VOAs associated to root lattices A_1^8 , D_4^2 and D_8 , respectively. Hence we obtain this lemma by Lemma 4.15 (1). \square

Lemma 4.21. *Let $\mathcal{S} = \mathcal{S}(5, k_1, 0, \varepsilon)$ be the maximal totally singular subspace of $R(V)^3$ given in Theorem 4.6. Let $\mathcal{T} = \{(0, t, \varphi(t)) \mid t \in T\}$.*

- (1) *The subspace $\mathfrak{V}(\mathcal{T})_1$ is an ideal of $\mathfrak{V}(\mathcal{S})_1$.*
- (2) *The rank of $\mathfrak{V}(\mathcal{T})_1$ is 2, 4, 8 if $\dim(T) = 4, 6, 8$, respectively.*

Proof. It is easy to see that \mathcal{T} satisfies $(M \boxtimes M')_1 = 0$ for all $[M] \in \mathcal{T}$, $[M'] \in \mathcal{S} \setminus \mathcal{T}$ with $M_1 \neq 0$ and $M'_1 \neq 0$. Then (1) follows from Lemma 4.15 (2).

Let T_0 be a maximal subset of $\overline{S(T)}$ such that $a + b \in S(T)$ for all distinct $a, b \in T_0$. Set $\mathcal{T}_0 = \{(0, t, \varphi(t)) \mid t \in T_0\}$. Then $\mathfrak{V}(\mathcal{T}_0)_1$ is a Cartan subalgebra of $\mathfrak{V}(\mathcal{T})_1$ by Lemmas 4.15 (3) and 4.16. It is easy to see that $|T_0| = 2, 4, 8$ if $\dim T = 4, 6, 8$. Hence we have (2). \square

Lemma 4.22. *Let $\mathcal{S} = \mathcal{S}(5, k_1, 1, \varepsilon)$ be the maximal totally singular subspace of $R(V)^3$ given in Theorem 4.6. Let H be a Cartan subalgebra of $\mathfrak{V}(\mathcal{S})_1$ contained in $\mathfrak{V}(\{(s_1, s_2, 0) \mid s_i \in S_i\})_1$. Let $\mathcal{T} = \{(0, t + s, \varphi(t)) \mid t \in T, s \in S_2\} \setminus \{(0, s, 0) \mid s \in S_2\}$. Then $\mathfrak{V}(\mathcal{T})_1$ is a sum of root spaces corresponding to a sum of irreducible components of root systems of $\mathfrak{V}(\mathcal{S})_1$.*

Proof. Clearly, H preserves $\mathfrak{V}(\mathcal{T})_1$. Hence $\mathfrak{V}(\mathcal{T})_1$ is a sum of root spaces.

Let $[M] \in \mathcal{S}$ with $M_1 \neq 0$. Then $(M \boxtimes M')_1 \neq 0$ for some $[M'] \in \mathcal{T}$ if and only if $[M] \in \mathcal{T} \cup \{(0, s, 0) \mid s \in S_2\}$. Hence we obtain this lemma. \square

Lemma 4.23. *Let $\mathcal{S} = \mathcal{S}(5, k_1, 0)$ be the maximal totally singular subspace of $R(V)^3$ given in Theorem 4.8. Let $\mathcal{T} = \{(0, t, \varphi(t)) \mid t \in T \perp \mathbb{F}_2 y\} \setminus \{(0, y, y)\}$, where $\varphi(y) = y$.*

(1) *The subspace $\mathfrak{V}(\mathcal{T})_1$ is an ideal of $\mathfrak{V}(\mathcal{S})_1$.*

(2) *Let $2m$ be the dimension of T . Then the rank of $\mathfrak{V}(\mathcal{T})_1$ is $2^m - 1$.*

Proof. It is easy to see that \mathcal{T} satisfies $(M \boxtimes M')_1 = 0$ for all $[M] \in \mathcal{T}$, $[M'] \in \mathcal{S} \setminus \mathcal{T}$ with $M_1 \neq 0$ and $M'_1 \neq 0$. Then (1) follows from Lemma 4.15 (2).

Let U be a maximal totally singular subspace of T . Then $\dim U = m$. Set $\mathcal{U} = \{(0, a + y, \varphi(a) + y) \mid a \in U\} \setminus \{(0, y, y)\}$ and $H = \mathfrak{V}(\mathcal{U})_1$. Then $M_1 \neq 0$ for all $[M] \in \mathcal{U}$. Moreover for $[M], [M'] \in \mathcal{U}$, $(M \boxtimes M')_1 = 0$. By Lemma 4.15 (3), H is abelian, and its dimension is $2^m - 1$. By Lemma 4.16, it is toral. Let us show that H is maximal.

Let $v \in \mathfrak{V}(\mathcal{T})_1$ with $v_{(0)}H = 0$. Let $v = \sum_{[M] \in \mathcal{T}} v_M$, where $v_M \in M_1$. Suppose that $v_M \neq 0$ for $[M] = (0, b + y, \varphi(b) + y)$, $b \in S(T)$. If $b \notin U$ then there exists $c \in U$ such that $\langle b, c \rangle = 1$, then $(v_M)_{(0)}M'_1 \neq 0$, where $[M'] = (0, c + y, \varphi(c) + y)$. It contradicts $v_{(0)}H = 0$. Hence $b \in U$. Suppose that $v_M \neq 0$ for $[M] = (0, b, \varphi(b))$, $b \in \overline{S(T)}$. Then there exists $c \in U$ such that $\langle b, c \rangle = 1$, then $(v_M)_{(0)}M' \neq 0$, where $[M'] = (0, c + y, \varphi(c) + y)$. This contradicts the assumption. Thus $v \in H$, and we obtain (2). \square

Later, we use the following lemma about holomorphic VOAs of central charge 16.

Lemma 4.24. (1) *The \mathbb{Z}_2 -orbifold of $V_{E_8 \oplus E_8}$ is isomorphic to $V_{D_{16}^+}$, and that of $V_{D_{16}^+}$ is isomorphic to $V_{E_8 \oplus E_8}$.*

(2) *Let $\mathcal{T} = \{(a, a) \mid a \in R(V)\}$. Then $\mathfrak{V}(\mathcal{T}) \cong V_{D_{16}^+}$.*

Proof. By [Sh06, Lemma 3.4] both $V_{E_8 \oplus E_8}^+$ and $V_{D_{16}^+}^+$ are isomorphic to V_N for some even lattice N such that $E_8 \oplus E_8$ and D_{16}^+ are even overlattices of N of index 2. Hence we obtain (1).

Let U_0 be a maximal totally singular subspace of $R(V)$. Let $\mathcal{T}_0 = \{(a, b) \mid a, b \in U_0\}$. Then $\mathfrak{V}(\mathcal{T}_0) \cong V_{E_8 \oplus E_8}$. Consider its \mathbb{Z}_2 -orbifold associated to the lift of $(-1, -1) \in$

$\text{Aut}(\sqrt{2}E_8 \oplus \sqrt{2}E_8)$. Then it is isomorphic to $\mathfrak{V}(\mathcal{T}_1)$, where $\mathcal{T}_1 = \{(a, b) \mid a, b \in U_1\} \oplus \{(a, a) \mid a \in U'_1\}$, where U_1 is a codimension 1 subspace of U and U'_1 is a complementary subspace of U_1 in U_1^\perp . By (1), $\mathfrak{V}(\mathcal{T}_1) \cong V_{D_{16}^+}$.

Let $\mathcal{T}_i = \{(a, b) \mid a, b \in U_i\} \oplus \{(a, a) \mid a \in U'_i\}$, where U_i is a codimension 1 subspace of U_{i-1} and U'_i is a complementary subspace of U_i in U_i^\perp . By the same argument, we obtain $\mathfrak{V}(\mathcal{T}_2) \cong V_{E_8 \oplus E_8}$, $\mathfrak{V}(\mathcal{T}_3) \cong V_{D_{16}^+}$, $\mathfrak{V}(\mathcal{T}_4) \cong V_{E_8 \oplus E_8}$, and $\mathfrak{V}(\mathcal{T}_5) \cong V_{D_{16}^+}$. Since $\mathcal{T}_5 = \mathcal{T}$, we obtain this lemma. \square

4.6. Determination of the Lie algebra structure of $\mathfrak{V}(\mathcal{S})$ for $\mathcal{S} \subset R(V)^3$. In this subsection, we determine the Lie algebra structure of the weight 1 subspace of $\mathfrak{V}(\mathcal{S})$ for $\mathcal{S} = \mathcal{S}(5, k_1, k_2, \varepsilon)$ and $\mathcal{S}(5, k_1, k_2)$.

First we will recall several important results from [DM04, DM06].

Proposition 4.25. [DM04, Theorem 3 and (3.6)] *Let V be a C_2 -cofinite holomorphic VOA of CFT type. Suppose the central charge of V is 24.*

- (a) *The Lie algebra V_1 has Lie rank less than or equal to 24 and is either abelian (including 0) or semisimple.*
- (b) *Suppose V_1 is semisimple, that is,*

$$V_1 = \mathfrak{g}_{1,k_1} \oplus \mathfrak{g}_{2,k_2} \oplus \cdots \oplus \mathfrak{g}_{n,k_n},$$

where \mathfrak{g}_i is a simple Lie algebra whose affine Lie algebra has level k_i on V . Then

$$(4.8) \quad \frac{h_i^\vee}{k_i} = \frac{(\dim V_1 - 24)}{24},$$

where h_i^\vee is the dual Coxeter number of \mathfrak{g}_i . In particular, the ratio h_i^\vee/k_i is independent of \mathfrak{g}_i .

Proposition 4.26. [DM06, Theorem 3.1] *Let V be a simple self-dual VOA which is C_2 -cofinite and of CFT type. Let \mathfrak{g} be a simple Lie subalgebra of V_1 , k the level of V as module for the corresponding affine Lie algebra. Then k is a positive integer.*

Remark 4.27. Let \mathcal{T} be a totally singular subspace of $R(V)^k$. Since $\mathfrak{V}(\mathcal{T})$ is framed, it is simple, rational, C_2 -cofinite and of CFT type ([DGH98]).

Case: $\mathcal{S} = \mathcal{S}(5, 1, 0, +)$. By Lemma 4.13 and Corollary 4.18, $\dim \mathfrak{V}(\mathcal{S})_1 = 60$. By Proposition 4.25, each simple component \mathfrak{g}_i of $\mathfrak{V}(\mathcal{S})_1$ satisfies $h_i/k_i = 3/2$. Hence by Proposition 4.26, \mathfrak{g}_i is one of the following.

Type	$A_{2,2}$	$A_{5,4}$	$C_{2,2}$	$B_{5,6}$	$C_{5,4}$	$D_{4,4}$	$F_{4,6}$
h^\vee	3	6	3	9	6	6	9
Dimension	8	35	10	55	55	28	52

By Lemma 4.21, $\mathfrak{V}(\mathcal{S})_1$ has a 28-dimensional ideal with rank 4. Hence $\mathfrak{V}(\mathcal{S})_1$ is isomorphic to $D_{4,4}(A_{2,2})^4$.

Proposition 4.28. *The Lie algebra structure of $\mathfrak{V}(\mathcal{S}(5, 1, 0, +))_1$ is $D_{4,4}(A_{2,2})^4$.*

Case: $\mathcal{S} = \mathcal{S}(5, 1, 0, -)$. By Lemma 4.13 and Corollary 4.18, $\dim \mathfrak{V}(\mathcal{S})_1 = 84$. By Proposition 4.25, each simple component \mathfrak{g}_i of $\mathfrak{V}(\mathcal{S})_1$ satisfies $h_i/k_i = 5/2$. Hence by Proposition 4.26, \mathfrak{g}_i is one of the following.

Type	$A_{4,2}$	$B_{3,2}$	$C_{4,2}$	$D_{6,4}$
h^\vee	5	5	5	10
Dimension	24	21	36	66

By Lemma 4.21, $\mathfrak{V}(\mathcal{S})_1$ has a 36-dimensional ideal with rank 4. Hence $\mathfrak{V}(\mathcal{S})_1$ must be isomorphic to $C_{4,2}(A_{4,2})^2$.

Proposition 4.29. *The Lie algebra structure of $\mathfrak{V}(\mathcal{S}(5, 1, 0, -))_1$ is $C_{4,2}(A_{4,2})^2$.*

Case: $\mathcal{S} = \mathcal{S}(5, 3, 0, +)$. By Lemma 4.13 and Corollary 4.18, $\dim \mathfrak{V}(\mathcal{S})_1 = 192$. By Proposition 4.25, each simple component \mathfrak{g}_i of $\mathfrak{V}(\mathcal{S})_1$ satisfies $h_i/k_i = 7$. Hence by Proposition 4.26, \mathfrak{g}_i is one of the following.

Type	$A_{6,1}$	$B_{4,1}$	$C_{6,1}$	$D_{8,2}$
h^\vee	7	7	7	14
Dimension	48	36	78	120

By Lemma 4.21, $\mathfrak{V}(\mathcal{S})_1$ has a 120-dimensional ideal with rank 8. Hence $\mathfrak{V}(\mathcal{S})_1$ must be isomorphic to $D_{8,2}(B_{4,1})^2$.

Proposition 4.30. *The Lie algebra structure of $\mathfrak{V}(\mathcal{S}(5, 3, 0, +))_1$ is $D_{8,2}(B_{4,1})^2$.*

Case: $\mathcal{S} = \mathcal{S}(5, 3, 0, -)$. By Lemma 4.13 and Corollary 4.18, $\dim \mathfrak{V}(\mathcal{S})_1 = 240$. By Proposition 4.25, each simple component \mathfrak{g}_i of $\mathfrak{V}(\mathcal{S})_1$ satisfies $h_i/k_i = 9$. Hence by Proposition 4.26, \mathfrak{g}_i is one of the following.

Type	$A_{8,1}$	$B_{5,1}$	$C_{8,1}$	$D_{10,2}$	$E_{7,2}$	$F_{4,1}$
h^\vee	9	9	9	18	18	9
Dimension	80	55	136	190	133	52

By Lemma 4.21, $\mathfrak{V}(\mathcal{S})_1$ has a 136-dimensional ideal with rank 8. Hence $\mathfrak{V}(\mathcal{S})_1$ must be isomorphic to $C_{8,1}(F_{4,1})^2$.

Proposition 4.31. *The Lie algebra structure of $\mathfrak{V}(\mathcal{S}(5, 3, 0, -))_1$ is $C_{8,1}(F_{4,1})^2$.*

Case: $\mathcal{S} = \mathcal{S}(5, 5, 0, +)$. By Lemma 4.13 and Corollary 4.18, $\dim \mathfrak{V}(\mathcal{S})_1 = 744$. By Proposition 4.25, each simple component \mathfrak{g}_i satisfies $h_i/k_i = 30$. Hence by Proposition 4.26, \mathfrak{g}_i is $D_{16,1}$ or $E_{8,1}$. Note that their dimensions are 496 and 248. Hence $\mathfrak{V}(\mathcal{S})_1$ is isomorphic to $(E_{8,1})^3$ or $E_{8,1}D_{16,1}$.

By the construction in Theorem 4.6, S_1 is a 5-dimensional totally singular space. Then we have $P = Q = 0$ and $T = U = R(V)$. Hence, $\mathcal{S}(5, 5, 0, +)$ is spanned by

$$\begin{aligned} (a, 0, 0), \quad a \in S_1, \\ (0, b, b), \quad b \in R(V). \end{aligned}$$

Since S_1 is a maximal totally singular subspace of $R(V)$, $\mathfrak{V}(\{(a, 0, 0) \mid a \in S_1\})$ is isomorphic to $V_{E_8} \otimes V^{\otimes 2}$. By Lemma 4.24, $\mathfrak{V}(\{(0, b, b) \mid b \in R(V)\})$ is isomorphic to $V \otimes V_{D_{16}^+}$. Hence we obtain the following.

Proposition 4.32. *The VOA $\mathfrak{V}(\mathcal{S}(5, 5, 0, +))$ is isomorphic to the lattice VOA associated to the Niemeier lattice $N(E_8D_{16})$.*

Case: $\mathcal{S} = \mathcal{S}(5, 2, 1, +)$. By Lemma 4.13 and Corollary 4.18, $\dim \mathfrak{V}(\mathcal{S})_1 = 120$. By Proposition 4.25, each simple component \mathfrak{g}_i of $\mathfrak{V}(\mathcal{S})_1$ satisfies $h_i/k_i = 4$. Hence by Proposition 4.26, \mathfrak{g}_i is one of the following.

Type	$A_{3,1}$	$A_{7,2}$	$C_{3,1}$	$C_{7,2}$	$D_{5,2}$	$D_{7,3}$	$E_{6,3}$	$G_{2,1}$
h^\vee	4	8	4	8	8	12	12	4
Dimension	15	63	21	105	45	91	78	14

By Lemma 4.22, $\mathfrak{V}(\mathcal{S})_1$ contains an ideal with 56-dimensional root space. Hence $\mathfrak{V}(\mathcal{S})_1$ is isomorphic to $A_{7,2}(C_{3,1})^2A_{3,1}$ or $A_{7,2}A_{3,1}(G_{2,1})^3$. By Lemma 4.20, $\mathfrak{V}(\mathcal{S})_1$ contains $(A_{1,1})^8$. Hence it must be isomorphic to $A_{7,2}(C_{3,1})^2A_{3,1}$.

Proposition 4.33. *The Lie algebra structure of $\mathfrak{V}(\mathcal{S}(5, 2, 1, +))_1$ is $A_{7,2}(C_{3,1})^2A_{3,1}$.*

Case: $\mathcal{S} = \mathcal{S}(5, 2, 1, -)$. By Lemma 4.13 and Corollary 4.18, $\dim \mathfrak{V}(\mathcal{S})_1 = 168$. By Proposition 4.25, each simple component \mathfrak{g}_i of $\mathfrak{V}(\mathcal{S})_1$ satisfies $h_i/k_i = 6$. Hence by Proposition 4.26, \mathfrak{g}_i is one of the following.

Type	$A_{5,1}$	$A_{11,2}$	$C_{5,1}$	$D_{4,1}$	$D_{7,2}$	$E_{6,2}$	$E_{7,3}$
h^\vee	6	12	6	6	12	12	18
Dimension	35	143	55	28	91	78	133

By Lemma 4.22, $\mathfrak{V}(\mathcal{S})_1$ contains an ideal with 72-dimensional root space. Hence $\mathfrak{V}(\mathcal{S})_1$ is isomorphic to $E_{6,2}C_{5,1}A_{5,1}$,

Proposition 4.34. *The Lie algebra structure of $\mathfrak{V}(\mathcal{S}(5, 2, 1, -))_1$ is $E_{6,2}C_{5,1}A_{5,1}$.*

Case: $\mathcal{S} = \mathcal{S}(5, 4, 1, +)$. By Lemma 4.13 and Corollary 4.18, $\dim \mathfrak{V}(\mathcal{S})_1 = 384$. By Proposition 4.25, each simple component \mathfrak{g}_i of $\mathfrak{V}(\mathcal{S})_1$ satisfies $h_i/k_i = 15$. Hence by Proposition 4.26, \mathfrak{g}_i is one of the following.

Type	$A_{14,1}$	$B_{8,1}$	$E_{8,2}$
h^\vee	15	15	30
Dimension	224	136	248

Hence $\mathfrak{V}(\mathcal{S})_1$ is isomorphic to $E_{8,2}B_{8,1}$.

Proposition 4.35. *The Lie algebra structure of $\mathfrak{V}(\mathcal{S}(5, 4, 1, +))_1$ is $E_{8,2}B_{8,1}$.*

Remark 4.36. In [Sc93], $E_{8,2}B_{8,1}$ was written as $E_{8,2}B_{6,1}$, which is a misprint.

Case: $\mathcal{S} = \mathcal{S}(5, 3, 2, +)$. By Lemma 4.13 and Corollary 4.18, $\dim \mathfrak{V}(\mathcal{S})_1 = 240$. By Proposition 4.25, each simple component \mathfrak{g}_i of $\mathfrak{V}(\mathcal{S})_1$ satisfies $h_i/k_i = 9$. Hence by Proposition 4.26, \mathfrak{g}_i is one of the following.

Type	$A_{8,1}$	$B_{5,1}$	$C_{8,1}$	$D_{10,2}$	$E_{7,2}$	$F_{4,1}$
h^\vee	9	9	9	18	18	9
Dimension	80	55	136	190	133	52

Hence $\mathfrak{V}(\mathcal{S})_1$ is isomorphic to $(A_{8,1})^3$, $C_{8,1}(F_{4,1})^2$ or $B_{5,1}E_{7,2}F_{4,1}$. By Lemma 4.19, the rank of $\mathfrak{V}(\mathcal{S})_1$ is 16. Moreover, by Lemma 4.20, $\mathfrak{V}(\mathcal{S})_1$ has a subalgebra $(D_{4,1})^2(A_{1,1})^8$. Hence $\mathfrak{V}(\mathcal{S})_1$ must be isomorphic to $C_{8,1}(F_{4,1})^2$.

Proposition 4.37. *The Lie algebra structure of $\mathfrak{V}(\mathcal{S}(5, 3, 2, +))_1$ is $C_{8,1}(F_{4,1})^2$.*

Case: $\mathcal{S} = \mathcal{S}(5, 0, 0)$. By Lemma 4.13 and Corollary 4.18, $\dim \mathfrak{V}(\mathcal{S})_1 = 48$. By Propositions 4.25, each simple component \mathfrak{g}_i of $\mathfrak{V}(\mathcal{S})_1$ satisfies $h_i/k_i = 1$.

We use the notation in Theorem 4.8. Set $\mathcal{T} = \text{Span}_{\mathbb{F}_2}\{(y, y, 0), (y, 0, y)\} \subset \mathcal{S}$. Then $\mathfrak{V}(\mathcal{T})_1$ is a 3-dimensional ideal of $\mathfrak{V}(\mathcal{S})_1$ isomorphic to a Lie algebra of type A_1 . Moreover, by Lemma 4.23, $\mathfrak{V}(\mathcal{S})_1$ has a 15-dimensional ideal

$$\mathfrak{V}(\{(a + y, a + y, 0), (b, b, 0) \mid a \in S(P)^\times, b \in \overline{S(P)}\})_1$$

of rank 3. By the same argument in the proof of Lemma 4.23,

$$\begin{aligned} &\mathfrak{V}(\{(a + y, 0, a + y), (b, 0, b) \mid a \in S(Q)^\times, b \in \overline{S(Q)}\})_1, \\ &\mathfrak{V}(\{(0, a + y, \varphi(a) + y), (0, b, \varphi(b)) \mid a \in S(T)^\times, b \in \overline{S(T)}\})_1, \end{aligned}$$

are 15-dimensional ideals of rank 3. Hence by Proposition 4.26, $\mathfrak{V}(\mathcal{S})_1$ is isomorphic to $(A_{3,4})^3 A_{1,2}$.

Proposition 4.38. *The Lie algebra structure of $\mathfrak{V}(\mathcal{S}(5, 0, 0))_1$ is $(A_{3,4})^3 A_{1,2}$.*

Case: $\mathcal{S} = \mathcal{S}(5, 2, 0)$. By Lemma 4.13 and Corollary 4.18, $\dim \mathfrak{V}(\mathcal{S})_1 = 120$. By Proposition 4.25, each simple component \mathfrak{g}_i of $\mathfrak{V}(\mathcal{S})_1$ satisfies $h_i/k_i = 4$. Hence by Proposition 4.26, \mathfrak{g}_i is one of the following.

Type	$A_{3,1}$	$A_{7,2}$	$C_{3,1}$	$C_{7,2}$	$D_{5,2}$	$D_{7,3}$	$E_{6,3}$	$G_{2,1}$
h^\vee	4	8	4	8	8	12	12	4
Dimension	15	63	21	105	45	91	78	14

By Lemma 4.23, $\mathfrak{V}(\mathcal{S})_1$ has a 63-dimensional ideal. Hence $\mathfrak{V}(\mathcal{S})_1$ must be isomorphic to $A_{7,2}(C_{3,1})^2 A_{3,1}$ or $A_{7,2} A_{3,1}(G_{2,1})^3$. By Lemma 4.20, the 57-dimensional ideal of $\mathfrak{V}(\mathcal{S})_1$ contains $(A_{1,1})^8$. Hence $\mathfrak{V}(\mathcal{S})_1$ must be isomorphic to $A_{7,2}(C_{3,1})^2 A_{3,1}$.

Proposition 4.39. *The Lie algebra structure of $\mathfrak{V}(\mathcal{S}(5, 2, 0))_1$ is $A_{7,2}(C_{3,1})^2 A_{3,1}$.*

Case: $\mathcal{S} = \mathcal{S}(5, 4, 0)$. By Lemma 4.13 and Corollary 4.18, $\dim \mathfrak{V}(\mathcal{S})_1 = 408$. By Proposition 4.25, each simple component \mathfrak{g}_i of $\mathfrak{V}(\mathcal{S})_1$ satisfies $h_i/k_i = 16$. Hence by Proposition 4.26, \mathfrak{g}_i is of type $A_{15,1}$ or $D_{9,1}$. Note that their dimensions are 255 and 153. Hence $\mathfrak{V}(\mathcal{S})_1$ is isomorphic to $A_{15,1} D_{9,1}$.

Proposition 4.40. *The VOA $\mathfrak{V}(\mathcal{S}(5, 4, 0))$ is isomorphic to the lattice VOA associated with the Niemeier lattice $N(A_{15} D_9)$.*

Case: $\mathcal{S} = \mathcal{S}(5, 1, 1)$. By Lemma 4.13 and Corollary 4.18, $\dim \mathfrak{V}(\mathcal{S})_1 = 96$. By Propositions 4.25, each simple component \mathfrak{g}_i of $\mathfrak{V}(\mathcal{S})_1$ satisfies $h_i/k_i = 3$. Hence by Proposition 4.26, \mathfrak{g}_i is one of the following.

Type	$A_{2,1}$	$A_{5,2}$	$A_{8,3}$	$C_{2,1}$	$B_{5,3}$	$C_{5,2}$	$D_{4,2}$	$D_{7,4}$	$E_{6,4}$	$F_{4,3}$
h^\vee	3	6	9	3	9	6	6	12	12	9
Dimension	8	35	80	10	55	55	28	91	78	52

Take non-zero $t_i \in S_i$. Then by Lemma 4.19, $H = \mathfrak{V}(\{(t_1, 0, 0), (0, t_2, 0)\})_1$ is a Cartan subalgebra of $\mathfrak{V}(\mathcal{S})_1$, and the rank is 16. Consider the root space decomposition with respect to H . Then it is easy to see that

$$\begin{aligned}
& \mathfrak{V}(\{(y + s_1, y + s_2, 0), (y + s_1, 0, y), (0, y + s_2, y) \mid s_i \in S_i\})_1, \\
& \mathfrak{V}(\{(y + a + s_1, y + a + s_2, 0), (b + s_1, b + s_2, 0) \mid s_i \in S_i, a \in S(P), b \in \overline{S(P)}\})_1, \\
& \mathfrak{V}(\{(y + a + s_1, 0, y + a), (b + s_1, 0, b) \mid s_1 \in S_1, a \in S(Q), b \in \overline{S(Q)}\})_1, \\
& \mathfrak{V}(\{(0, y + a + s_2, y + \varphi(a)), (0, b + s_2, \varphi(b)) \mid s_2 \in S_2, a \in S(T), b \in \overline{S(T)}\})_1
\end{aligned}$$

are mutually orthogonal root spaces. Here we use the same notations as in Theorem 4.8. Note that their dimensions are 8, 12, 30, 30. Hence the 8-dimensional and 12-dimensional root spaces are $C_{2,1}$ and $(A_{2,1})^2$, respectively. Note that the 30-dimensional root space is $(A_{2,1})^5$, $D_{4,2}A_{2,1}$ or $A_{5,2}$. Since the rank of $\mathfrak{V}(\mathcal{S})_1$ is 16, $\mathfrak{V}(\mathcal{S})_1$ must be isomorphic to $(A_{5,2})^2 C_{2,1} (A_{2,1})^2$.

Proposition 4.41. *The Lie algebra structure of $\mathfrak{V}(\mathcal{S}(5, 1, 1))_1$ is $(A_{5,2})^2 C_{2,1} (A_{2,1})^2$.*

Case: $\mathcal{S} = \mathcal{S}(5, 3, 1)$. By Lemma 4.13 and Corollary 4.18, $\dim \mathfrak{V}(\mathcal{S})_1 = 240$. By Proposition 4.25, each simple component \mathfrak{g}_i of $\mathfrak{V}(\mathcal{S})_1$ satisfies $h_i/k_i = 9$. Hence by Proposition 4.26, \mathfrak{g}_i is one of the following.

Type	$A_{8,1}$	$B_{5,1}$	$C_{8,1}$	$D_{10,2}$	$E_{7,2}$	$F_{4,1}$
h^\vee	9	9	9	18	18	9
Dimension	80	55	136	190	133	52

Take non-zero $t_i \in S_i$. Then by Lemma 4.19, $\mathfrak{V}(\{(t_1, 0, 0), (0, t_2, 0)\})_1$ is a Cartan subalgebra of $\mathfrak{V}(\mathcal{S})_1$, and the rank is 16. By the similar arguments as in Lemma 4.22, $\mathfrak{V}(\mathcal{S})_1$ has rank 16 and has an ideal

$$\mathfrak{V}(\{(0, y + a + s_2, y + \varphi(a)), (0, b + s_2, \varphi(b)) \mid s_2 \in S_2, a \in S(T), b \in \overline{S(T)}\})_1$$

with 126-dimensional root space. Hence $\mathfrak{V}(\mathcal{S})_1$ contains $E_{7,2}$ and the Lie algebra structure of $\mathfrak{V}(\mathcal{S})_1$ is isomorphic to $E_{7,2} B_{5,1} F_{4,1}$.

Proposition 4.42. *The Lie algebra structure of $\mathfrak{V}(\mathcal{S}(5, 3, 1))_1$ is $E_{7,2} B_{5,1} F_{4,1}$.*

Case: $\mathcal{S} = \mathcal{S}(5, 2, 2)$. By Lemma 4.13 and Corollary 4.18, $\dim \mathfrak{V}(\mathcal{S})_1 = 192$. By Proposition 4.25, each simple component \mathfrak{g}_i of $\mathfrak{V}(\mathcal{S})_1$ satisfies $h_i/k_i = 7$. Hence by Proposition 4.26, \mathfrak{g}_i is one of the following.

Type	$A_{6,1}$	$B_{4,1}$	$C_{6,1}$	$D_{8,2}$
h^\vee	7	7	7	14
Dimension	48	36	78	120

By Lemma 4.19, the rank is 16. Hence $\mathfrak{V}(\mathcal{S})_1$ must be isomorphic to $D_{8,2}(B_{4,1})^2$ or $(C_{6,1})^2 B_{4,1}$. Moreover, by Lemma 4.20, $\mathfrak{V}(\mathcal{S})_1$ has a subalgebra of type $(A_{1,1})^{16}$. Hence it must be isomorphic to $(C_{6,1})^2 B_{4,1}$.

Proposition 4.43. *The Lie algebra structure of $\mathfrak{V}(\mathcal{S}(5, 2, 2))_1$ is $(C_{6,1})^2 B_{4,1}$.*

4.7. Isomorphism type of the VOA $\mathfrak{V}(\mathcal{S}(5, 3, 0, +))$. In this subsection, we will show that the VOA $\mathfrak{V}(\mathcal{S}(5, 3, 0, +))$ is isomorphic to $\tilde{V}_{N(A_{15}D_9)}$. In order to do it, we use the \mathbb{Z}_2 -orbifolds of VOAs associated to maximal totally singular subspaces of $R(V)^3$.

Let \mathcal{S} be a maximal totally singular subspace of $R(V)^3$. Then $\mathfrak{V}(\mathcal{S})$ is a holomorphic VOA. Let $W \in R(V)^3 \setminus \mathcal{S}$ with $q_V^3(W) = 0$. Let $\chi_W : \mathcal{S} \rightarrow \mathbb{Z}_2$ be the character of \mathcal{S} defined by $\chi_W(W') = \langle W, W' \rangle$. Then χ_W induces the automorphism g_W of $\mathfrak{V}(\mathcal{S})$ acting on M' by $(-1)^{\chi_W(W')}$ for $W' = [M'] \in \mathcal{S}$.

Proposition 4.44. *The \mathbb{Z}_2 -orbifold of $\mathfrak{V}(\mathcal{S})$ associated to g_W is given by $\mathfrak{V}(\text{Span}_{\mathbb{F}_2}\{W, \mathcal{S} \cap W^\perp\})$.*

Proof. The subspace fixed by g_W is $\mathfrak{V}(\mathcal{S} \cap W^\perp)$. By the maximality of \mathcal{S} , the irreducible modules for $\mathfrak{V}(\mathcal{S} \cap W^\perp)$ with integral weights are $\mathfrak{V}(\mathcal{S} \cap W^\perp)$, $\mathfrak{V}(\mathcal{S} \setminus W^\perp)$ and $\mathfrak{V}(W + (\mathcal{S} \cap W^\perp))$. Hence the \mathbb{Z}_2 -orbifold of $\mathfrak{V}(\mathcal{S})$ associated to g_W is

$$\mathfrak{V}(\mathcal{S} \cap W^\perp) \oplus \mathfrak{V}(W + (\mathcal{S} \cap W^\perp)) = \mathfrak{V}(\text{Span}_{\mathbb{F}_2}\{W, \mathcal{S} \cap W^\perp\})$$

as desired. \square

Let us consider $\mathcal{S} = \mathcal{S}(5, 4, 0)$. Then $\mathcal{S} = \text{Span}_{\mathbb{F}_2}\{(s, 0, 0), (0, t, t), (y, y, 0), (y, 0, y), (z, z, z) \mid s \in S_1, t \in T\}$, where T is an 8-dimensional non-singular quadratic subspace of $R(V)$, $\text{Span}_{\mathbb{F}_2}\{y, z\}$ is the orthogonal complement of T in $R(V)$, and S_1 is a maximal totally singular subspace of T (see Theorem 4.8). Note that $q_V(y) = 1$, $q_V(z) = 0$ and $\langle y, z \rangle = 1$. Take $s_0 \in S_1$ and $t_0 \in T$ with $q_V(t_0) = 0$ and $\langle s_0, t_0 \rangle = 1$. Set $W = (t_0, 0, z)$. Then $\text{Span}_{\mathbb{F}_2}\{W, \mathcal{S} \cap W^\perp\} = \text{Span}_{\mathbb{F}_2}\{(s, 0, 0), (0, t, t), (y, y, 0), (t_0, 0, z), (t_0 + z, z, 0), (s_0 + y, 0, y) \mid s \in S_1 \cap t_0^\perp, t \in T\}$. Since for $i = 1, 2$ and $j = 1, 3$, $\rho_i(\text{Span}_{\mathbb{F}_2}\{(y, y, 0), (t_0 + z, z, 0)\})$ and $\rho_j(\text{Span}_{\mathbb{F}_2}\{(t_0, 0, z), (s_0 + y, 0, y)\})$ are non-singular 2-dimensional quadratic subspaces of plus type, $\text{Span}_{\mathbb{F}_2}\{W, \mathcal{S} \cap W^\perp\}$ is conjugate to $\mathcal{S}(5, 3, 0, +)$ under $\text{Aut}(V) \wr \text{Sym}_3$. Hence by Proposition 4.44, $\mathfrak{V}(\mathcal{S}(5, 3, 0, +))$ is obtained by the \mathbb{Z}_2 -orbifold of $\mathfrak{V}(\mathcal{S}(5, 4, 0))$ associated to g_W .

Recall from Proposition 4.40 that $\mathfrak{V}(\mathcal{S}(5, 4, 0))$ is isomorphic to the VOA associated to the Niemeier lattice $N(A_{15,1}D_{9,1})$. Let us show that g_W is conjugate to a lift of the -1 -isometry of the lattice $N(A_{15,1}D_{9,1})$. By [DGH98, Appendix D], it suffices to show that g_W acts by -1 on a Cartan subalgebra of $\mathfrak{V}(\mathcal{S}(5, 4, 0))_1$. Consider the subspace $\mathfrak{V}(\{(s_0, 0, 0), (0, s + y, s + y) \mid s \in S_1\})_1$ of $\mathfrak{V}(\mathcal{S}(5, 4, 0))_1$. Then by Lemmas 4.16 and 4.23, it is a 24-dimensional toral abelian subalgebra, that is, a Cartan subalgebra. Since $\{(s_0, 0, 0), (0, s + y, s + y) \mid s \in S_1\} \cap W^\perp = \emptyset$, g_W acts by -1 on this Cartan subalgebra. Thus we obtain the following proposition.

Proposition 4.45. *The VOA $\mathfrak{V}(\mathcal{S}(5, 3, 0, +))$ is isomorphic to $\tilde{V}_{N(A_{15}D_9)}$.*

Classification of Lie algebra structures. By Lemmas 1.4 and 4.3, Theorem 4.11, Proposition 4.2 and Section 4.6, we obtain the following theorem.

Theorem 4.46. *Let U be a holomorphic simple current extension of $(V_{\sqrt{2}E_8}^+)^{\otimes 3}$. Then one of the following holds:*

- (1) U is isomorphic to a lattice VOA V_N or its \mathbb{Z}_2 -orbifold \tilde{V}_N for some even unimodular lattice N ;
- (2) The weight one subspace U_1 is isomorphic to one of the Lie algebras in Table 4.

TABLE 4. Lie algebra structure of $\mathfrak{V}(\mathcal{S})_1$ for $\mathcal{S} \subset R(V)^3$

\mathcal{S}	$\dim \mathfrak{V}(\mathcal{S})_1$	Lie algebra	No. in [Sc93]	Ref.
$\mathcal{S}(5, 1, 0, +)$	60	$D_{4,4}(A_{2,2})^4$	13	New
$\mathcal{S}(5, 1, 0, -)$	84	$C_{4,2}(A_{4,2})^2$	22	New
$\mathcal{S}(5, 3, 0, +)$	192	$D_{8,2}(B_{4,1})^2$	47	$\tilde{V}_{N(A_{15}D_9)}$
$\mathcal{S}(5, 3, 0, -)$	240	$C_{8,1}(F_{4,1})^2$	52	New
$\mathcal{S}(5, 5, 0, +)$	744	$D_{16,1}E_{8,1}$	69	$V_{N(D_{16}E_8)}$
$\mathcal{S}(5, 2, 1, +)$	120	$A_{7,2}(C_{3,1})^2A_{3,1}$	33	[La11]
$\mathcal{S}(5, 2, 1, -)$	168	$E_{6,2}C_{5,1}A_{5,1}$	44	New
$\mathcal{S}(5, 4, 1, +)$	384	$E_{8,2}B_{8,1}$	62	New
$\mathcal{S}(5, 3, 2, +)$	240	$C_{8,1}(F_{4,1})^2$	52	New
$\mathcal{S}(5, 0, 0)$	48	$(A_{3,4})^3A_{1,2}$	7	[La11]
$\mathcal{S}(5, 2, 0)$	120	$A_{7,2}(C_{3,1})^2A_{3,1}$	33	[La11]
$\mathcal{S}(5, 4, 0)$	408	$A_{15,1}D_{9,1}$	63	$V_{N(A_{15}D_9)}$
$\mathcal{S}(5, 1, 1)$	96	$(A_{5,2})^2C_{2,1}(A_{2,1})^2$	26	[La11]
$\mathcal{S}(5, 3, 1)$	240	$E_{7,2}B_{5,1}F_{4,1}$	53	New
$\mathcal{S}(5, 2, 2)$	192	$(C_{6,1})^2B_{4,1}$	48	[La11]

5. FRAMED VOAs ASSOCIATED TO SUBCODES OF $\mathcal{D}(e_8) \oplus \mathcal{D}(d_{16}^+)$

Recall that $\mathcal{D}(d_{16}^+)^\perp = \text{Span}_{\mathbb{Z}_2}\{d(\mathcal{E}_{16}), \ell(d_{16}^+)\}$ and the corresponding binary code VOA is isomorphic to $V_{\sqrt{2}D_{16}}^+$. Note also that $\mathcal{D}(e_8)^\perp \cong \text{RM}(2, 4)$ and $M_{\text{RM}(2,4)} \cong V_{\sqrt{2}E_8}^+$.

Throughout this section, let $V = V_{\sqrt{2}E_8}^+$ and $X = V_{\sqrt{2}D_{16}}^+$. For the detail of V and X , see Sections 3.1 and 3.2, respectively. In this section, we study holomorphic VOAs associated to maximal totally singular subspaces of $(R(X) \oplus R(V), q_X + q_V)$, which are holomorphic simple current extensions of $X \otimes V$, and classify such VOAs.

5.1. Simple current extensions of $V_{\sqrt{2}D_{16}}^+ \otimes V_{\sqrt{2}E_8}^+$. In this section, we study relations between simple current extensions of $X \otimes V$ and totally singular subspaces of $R(X) \oplus R(V)$.

We identify $R(X \otimes V)$ with $R(X) \oplus R(V)$ by Lemma 1.3. By Lemmas 3.3 (2) and 3.6 (2), $(R(X) \oplus R(V), q_X + q_V)$ is a non-singular 28-dimensional quadratic space of plus type over \mathbb{F}_2 .

Notation 5.1. Let \mathcal{T} be a subset of $R(X) \oplus R(V)$. Define $\mathfrak{V}(\mathcal{T}) = \oplus_{[M] \in \mathcal{T}} M$.

The following proposition can be shown by the same argument in [Sh11] (cf. Proposition 4.2).

Proposition 5.2. *Let $V = V_{\sqrt{2}E_8}^+$ and $X = V_{\sqrt{2}D_{16}}^+$. Then the $X \otimes V$ -module $\mathfrak{V}(\mathcal{T}) = \oplus_{[M] \in \mathcal{T}} M$ has a simple VOA structure which extends its $X \otimes V$ -module structure if and only if \mathcal{T} is a totally singular subspace of $R(X) \oplus R(V)$. Moreover, $\mathfrak{V}(\mathcal{T})$ is holomorphic if and only if \mathcal{T} is maximal.*

Remark 5.3. Let \mathcal{T} be a totally singular subspace of $R(X) \oplus R(V)$. Since V and X are framed, so is $\mathfrak{V}(\mathcal{T})$. Hence $\mathfrak{V}(\mathcal{T})$ is simple, rational, C_2 -cofinite and of CFT type ([DGH98]).

Clearly $\text{Aut}(X \otimes V)$ contains $\text{Aut}(X) \times \text{Aut}(V)$. By Lemma 1.4, conjugates of $\mathfrak{V}(\mathcal{T})$ under $\text{Aut}(X \otimes V)$ give isomorphic VOAs.

Lemma 5.4. *Let \mathcal{S} be a maximal totally singular subspace of $R(X) \oplus R(V)$.*

- (1) *If \mathcal{S} contains $(a_1, 0), (0, a_2)$ for some $a_1 \in \{[0]^{-}, [\alpha_1]^{\pm}\} \subset R(X)$ and $a_2 \in S(R(V))^{\times}$ then $\mathfrak{V}(\mathcal{S})$ is isomorphic to a lattice VOA V_L .*
- (2) *If \mathcal{S} contains (a_1, a_2) for some $a_1 \in \{[0]^{-}, [\alpha_1]^{\pm}\} \subset R(X)$ and $a_2 \in S(R(V))^{\times}$ then $\mathfrak{V}(\mathcal{S})$ is isomorphic to V_L or its \mathbb{Z}_2 -orbifold \tilde{V}_L .*
- (3) *If \mathcal{S} contains $([\alpha_c/2]^{\varepsilon}, 0)$ with $\text{wt}(c) = 8$ then $\mathfrak{V}(\mathcal{S})$ contains a full subVOA isomorphic to $(V_{\sqrt{2}E_8}^+)^{\otimes 3}$.*

Proof. Recall that the orbit of $[0]^{-}$ in $\text{Aut}(X)$ is $\{[0]^{-}, [\alpha_1]^{\pm}\}$ (see Table 3). Hence (1) and (2) are shown by the same argument in Lemma 4.4.

If \mathcal{S} contains $[\alpha_c/2]^{\varepsilon}$ with $\text{wt}(c) = 8$ then up to conjugation, we may assume that $([\alpha_c/2]^+, 0) \in \mathcal{S}$. Hence $\mathfrak{V}(\mathcal{S})$ contains a full subVOA isomorphic to $V_{\sqrt{2}E_8 \oplus \sqrt{2}E_8}^+ \otimes V_{\sqrt{2}E_8}^+$ by Lemma 3.5, which proves (3). \square

Lemma 5.5. *Let \mathcal{T} be a totally singular subspace of $R(X) \oplus R(V)$ and $[M]$ an element in \mathcal{T} with $M_1 \neq 0$. Then for any $a \in M_1$, $a_{(0)}$ is semisimple on $\mathfrak{V}(\mathcal{T})_1$.*

Proof. By the action of $\text{Aut}(X \otimes V)$ (cf. Table 3), we may assume that $[M]$ is the tensor product of irreducible modules of untwisted type for X and V . Hence this lemma follows from [FLM88]. \square

5.2. Classification of maximal totally singular subspaces of $R(X) \oplus R(Y)$. In this subsection, we study maximal totally singular subspaces of $R(X) \oplus R(V)$. Let ρ_1 and ρ_2 denote the projections from $R(X) \oplus R(V)$ to $R(X)$ and to $R(V)$, respectively. For a subset \mathcal{S} of $R(X) \oplus R(V)$, let $\mathcal{S}^{(i)} = \{W \in \mathcal{S} \mid \rho_i(W) = 0\}$.

Lemma 5.6. *Let \mathcal{S} be a maximal totally singular subspace of $R(X) \oplus R(V)$. Then the following hold:*

- (1) For $\{i, j\} = \{1, 2\}$, $\rho_i(\mathcal{S}) = \rho_i(\mathcal{S}^{(j)})^\perp$;
- (2) There is a bijection from $\rho_1(\mathcal{S})/\rho_1(\mathcal{S}^{(2)})$ to $\rho_2(\mathcal{S})/\rho_2(\mathcal{S}^{(1)})$;
- (3) $\dim \rho_1(\mathcal{S}^{(2)}) \geq 4$.

Proof. (1) follows from the maximality of \mathcal{S} .

Let $W^1 \in \rho_1(\mathcal{S})$. Then there is $W^2 \in \rho_2(\mathcal{S})$ such that $(W^1, W^2) \in \mathcal{S}$. By the definition of ρ_i , the map $W_1 + \rho_1(\mathcal{S}^{(2)}) \mapsto W_2 + \rho_2(\mathcal{S}^{(1)})$ is a well-defined bijection, which proves (2).

Since \mathcal{S} is maximal, $\dim \mathcal{S} = 14$. (3) follows from $\dim R(V) = 10$. \square

Proposition 5.7. *Let \mathcal{S} be a maximal totally singular subspace of $R(X) \oplus R(V)$. Assume that $\dim \rho_1(\mathcal{S}^{(2)}) \geq 5$. Then one of the following holds:*

- (1) $\mathfrak{V}(\mathcal{S})$ contains a full subVOA isomorphic to $(V_{\sqrt{2}E_8}^+)^{\otimes 3}$;
- (2) $\mathfrak{V}(\mathcal{S})$ is isomorphic to a lattice VOA or its \mathbb{Z}_2 -orbifold;
- (3) $\rho_1(\mathcal{S}^{(2)})$ is conjugate to

$$\text{Span}_{\mathbb{F}_2} \{[\alpha_{(1^4 0^{12})}/2]^+, [\alpha_{(1^2 0^2 1^2 0^{10})}/2]^+, [\alpha_{((10)^4 0^8)}/2]^+, [\alpha_{(1^{16})}/4]^+, [\chi_0]^+\}.$$

Proof. Set $T = \rho_1(\mathcal{S}^{(2)})$ and $d = \dim T$. Note that T is a totally singular subspace of $R(X)$. By the assumption, $\dim \rho_2(\mathcal{S}) = 14 - d \leq 9$. Hence $\dim \mathcal{S}^{(1)} \geq 1$. Up to the action of $\text{Aut}(V_{\sqrt{2}E_8}^+)$ on the second coordinate, we may assume that \mathcal{S} contains $([0]^+, [0]^-)$.

Set $T_1 = \{[\lambda]^\pm \mid \lambda \in (\sqrt{2}D_{16}^+)^*/\sqrt{2}D_{16}^+\} \cap T$. Then by Proposition 3.1, T_1 is a subspace of T , and $\dim T/T_1 \leq 1$, that is, $\dim T_1 \geq d - 1$. If T contains $[0]^-$ then (2) holds by the first paragraph and Lemma 5.4 (1). Hence, we may assume T does not contain $[0]^-$.

Let L be the overlattice of $\sqrt{2}D_{16}^+$ such that $T_1 = \{[\lambda]^\varepsilon \mid \lambda \in L/\sqrt{2}D_{16}^+\}$. Since T_1 is totally singular, L is even. It follows from $\dim T_1 \geq d - 1$ that $|L/\sqrt{2}D_{16}^+| \geq 2^{d-1}$. We now use the descriptions of $\sqrt{2}D_{16}^+$ and its dual lattice given in Section 3.2. Then L contains a sublattice

$$L_1 = \sum_{1 \leq i, j \leq 16} \mathbb{Z}(\alpha_i + \alpha_j) + \sum_{c \in C} \mathbb{Z}(\frac{\alpha_c}{2} - \delta_c \alpha_1)$$

such that $L/L_1 \subset \text{Span}_{\mathbb{F}_2}\{\alpha_1, \alpha_{(1^{16})}/4 - \alpha_d/2\}$, where $\delta_c \in \{0, 1\}$, $d \in \mathcal{E}_{16}$ and C is a doubly even code. If $L/\sqrt{2}D_{16}^+$ contains $\alpha_1 + \sqrt{2}D_{16}^+$ then up to the action of $\text{Aut}(V_{\sqrt{2}D_{16}^+}^+)$ on T , T contains $[0]^-$ by Table 3, and (2) holds. Hence we may assume that $|L/L_1| \leq 2$, which implies $|L_1/\sqrt{2}D_{16}^+| \geq 2^{d-2}$, namely, $\dim C \geq d - 1$. If $\dim C \geq 5$ then C has a weight 8 codeword, and by Lemma 5.4 (3), (1) holds. In particular, if $d \geq 6$ then (1) or (3) holds.

Assume $d = 5$. If $|T/T_1| = 1$ or $|L/L_1| = 1$ then $\dim C \geq 6$, and (1) holds. Hence we may assume that $T/T_1 = \{[0]^+, [\chi_\lambda]^+\}$ and that $L/L_1 = \{0, \alpha_{(1^{16})}/4 - \alpha_d/2\}$. By Lemma 5.4, we also may assume that C does not have weight 8 codewords. Hence C is equivalent to $\text{Span}_{\mathbb{F}_2}\{(1^4 0^{14}), (1^2 0^2 1^2 0^{10}), ((10)^4 0^8), (1^{16})\}$. Up to the action of lifts of $\text{Aut}(\sqrt{2}D_{16}^+)$ to $\text{Aut}(V_{\sqrt{2}D_{16}^+}^+)$, T is conjugate to

$$\text{Span}_{\mathbb{F}_2}\{[\alpha_{(1^4 0^{12})}/2]^+, [\alpha_{(1^2 0^2 1^2 0^{10})}/2]^+, [\alpha_{((10)^4 0^8)}/2]^+, [\alpha_{(1^{16})}/4]^+, [\chi_\lambda]^+\}.$$

Since T is totally singular, $\langle \lambda, \lambda \rangle + \langle v + \lambda, v + \lambda \rangle \in 2\mathbb{Z}$ for all $v \in L$ by Proposition 3.1. Hence $\langle \lambda, v \rangle \in \mathbb{Z}$ for all $v \in L$. Up to the action of $\langle \lambda, \cdot \rangle \in \text{Hom}(\sqrt{2}D_{16}^+, \mathbb{Z}_2) \subset \text{Aut}(V_{\sqrt{2}D_{16}^+}^+)$, we may assume that $\lambda = 0$. Hence (3) holds. \square

Proposition 5.8. *Let \mathcal{S} be a maximal totally singular subspace of $R(X) \oplus R(V)$. Assume that $\dim \rho_1(\mathcal{S}^{(2)}) = 4$. Then one of the following holds:*

- (1) V contains a full subVOA isomorphic to $(V_{\sqrt{2}E_8}^+)^{\otimes 3}$;
- (2) V is isomorphic to a lattice VOA or its \mathbb{Z}_2 -orbifold;
- (3) $\rho_1(\mathcal{S}^{(2)})$ is conjugate to $\text{Span}_{\mathbb{F}_2}\{[0]^-, [\alpha_1]^+, [\alpha_{(1^4 0^{12})}/2]^+, [\alpha_{(1^2 0^2 1^2 0^{10})}/2]^+\}$;
- (4) $\rho_1(\mathcal{S}^{(2)})$ is conjugate to $\text{Span}_{\mathbb{F}_2}\{[0]^-, [\alpha_{(1^4 0^{12})}/2]^+, [\alpha_{(1^2 0^2 1^2 0^{10})}/2]^+, [\alpha_{((10)^4 0^8)}/2]^+\}$;
- (5) $\rho_1(\mathcal{S}^{(2)})$ is conjugate to $\text{Span}_{\mathbb{F}_2}\{[0]^-, [\alpha_{(1^4 0^{12})}/2]^+, [\alpha_{(1^2 0^2 1^2 0^{10})}/2]^+, [\alpha_{(1^{16})}/4]^+\}$;
- (6) $\rho_1(\mathcal{S}^{(2)})$ is conjugate to $\text{Span}_{\mathbb{F}_2}\{[\alpha_{(1^4 0^{12})}/2]^+, [\alpha_{(1^2 0^2 1^2 0^{10})}/2]^+, [\alpha_{(1^{16})}/4]^+, [\chi_0]^+\}$.

Proof. Set $T = \rho_1(\mathcal{S}^{(2)})$. If T contains $[\alpha_c/2]^\pm$ with $\text{wt}(c) = 8$ then (1) holds by Lemma 5.4 (3). Hence we may assume that T does not contain such elements. If T contains neither $[0]^-$ nor $[\chi_\lambda]^+$ then $T^\perp \setminus T$ contains $[0]^-$ by Lemmas 3.2 and 5.6 (1). In this case, (2) holds up to conjugation. Hence we may assume that T contains $[0]^-$ or $[\chi_\lambda]^+$.

Assume that T contains $[0]^-$. Then by Lemma 3.2 T is a subset of

$$\{[0]^\pm, [\alpha_1]^\pm, [\alpha_c/2]^\pm, [\alpha_c/2 - \alpha_i]^\pm, [\alpha_{(1^{16})}/4 - \alpha_d/2]^\pm \mid \text{wt}(c) = 4, d \in \mathcal{E}_{16}\}.$$

By Lemma 3.2, T can not contain both $[\alpha_1]^\pm$ and $[\alpha_{(1^{16})}/4 - \alpha_d/2]^\pm$. Thus we obtain one of (3), (4) and (5) by Proposition 3.1 and Table 3.

Assume that T contains $[\chi_\lambda]^+$. Up to the action of $\text{Aut}(V_{\sqrt{2}D_{16}^+}^+)$, we may assume that $\lambda = 0$. If T does not contain $[\alpha_{(1^{16})}/4 - \alpha_d/2]^\pm$ then $g \circ T$ does not contain $[\chi_\lambda]^+$ for

some $g \in \text{Aut}(V_{\sqrt{2}D_{16}^+}^+)$ by Proposition 3.1 and Table 3, which is the case in the previous paragraph. Hence we may assume that T contains $[\alpha_{(1^{16})}/4 - \alpha_d/2]^+$. By the action of automorphisms induced from $\text{Aut}(\sqrt{2}D_{16}^+)$, we may assume $d = 0$. By Lemma 3.2, T does not contain $[\alpha_1]^+$. Thus we have (6). \square

Let us calculate the dimension of the weight 1 subspace of $\mathfrak{V}(\mathcal{S})$.

Proposition 5.9. *Let \mathcal{S} be a maximal totally singular subspace of $R(X) \oplus R(V)$. Then the dimension of $\mathfrak{V}(\mathcal{S})_1$ is given by*

$$(5.1) \quad \begin{aligned} & 16 \times |\rho_1(\mathcal{S}^{(2)}) \cap \{[0]^-, [\alpha_1]^\pm\}| + 4 \times |\rho_1(\mathcal{S}^{(2)}) \cap \{[\alpha_c/2]^\pm, [\alpha_c/2 - \alpha_1]^\pm \mid \text{wt}(c) = 4\}| \\ & + |\rho_1(\mathcal{S}^{(2)}) \cap \{[\alpha_{(1^{16})}/4 - \alpha_d/2]^\pm, [\chi_\lambda]^+ \mid d \in \mathcal{E}_{16}\}| + 8 \times (|\rho_2(\mathcal{S}^{(1)})| - 1) \\ & + |\rho_1(\mathcal{S}) \cap \{[\alpha_c/2 - \alpha_1]^\pm, [\alpha_c/2]^\pm \mid \text{wt}(c) = 2\}| \times |\rho_2(\mathcal{S}^{(1)})|. \end{aligned}$$

Proof. Let $(W^1, W^2) \in \mathcal{S}$. Then the lowest weights of W^i are non-negative half integers, and the sum of the lowest weights is an integer. Hence $\mathfrak{V}(\{(W^1, W^2)\})_1 \neq 0$ if and only if the lowest weights of both W^1 and W^2 are $1/2$, or the lowest weight of W^i is 1 and $W^j = [0]^+$, where $\{i, j\} = \{1, 2\}$. By Table 3, Lemmas 3.3 (2) and 5.6, the dimension of the weight 1 subspace of $\mathfrak{V}(\mathcal{S})$ is given by (5.1). \square

5.3. Determination of the Lie algebra structure of $\mathfrak{V}(\mathcal{S})_1$ for $\mathcal{S} \subset R(X) \oplus R(V)$.

In this subsection, we determine the Lie algebra structure of $\mathfrak{V}(\mathcal{S})_1$ for a maximal totally singular subspace \mathcal{S} of $R(X) \oplus R(V)$ described in Propositions 5.7 (3) and 5.8 (3)–(6).

Case: \mathcal{S} as in Propositions 5.7 (3).

Proposition 5.10. *Let \mathcal{S} be a maximal totally singular subspace of $R(X) \oplus R(V)$. Assume that $\rho_1(\mathcal{S}^{(2)}) = \text{Span}_{\mathbb{F}_2}\{[\alpha_{(1^4 0^{12})}/2]^+, [\alpha_{(1^2 0^2 1^2 0^{10})}/2]^+, [\alpha_{((10)^4 0^8)}/2]^+, [\alpha_{(1^{16})}/4]^+, [\chi_0]^+\}$. Set $D = \text{Span}_{\mathbb{F}_2}\{(1^4 0^{12}), (1^2 0^2 1^2 0^{10}), ((10)^4 0^8)\}$. Then the following hold:*

- (1) $\rho_1(\mathcal{S}) \cap \{[\alpha_c/2 - \alpha_1]^\pm, [\alpha_c/2]^\pm \mid \text{wt}(c) = 2\} = \{[\alpha_c/2 - \alpha_1]^+ \mid \text{wt}(c) = 2, c \in D^\perp\}$ and its size is 36;
- (2) $\dim \mathfrak{V}(\mathcal{S})_1 = 132$, and the Lie algebra structure of $\mathfrak{V}(\mathcal{S})_1$ is $A_{8,2}F_{4,2}$.

Proof. Note that the number of codewords in D^\perp with weight 2 is $\binom{9}{2} = 36$. (1) is easily calculated by Lemmas 3.2 and 5.6 (1).

By Proposition 5.9, we obtain

$$\dim \mathfrak{V}(\mathcal{S})_1 = 16 \times 0 + 4 \times 7 + 24 + 8 \times 1 + 36 \times 2 = 132.$$

By Proposition 4.25, each simple component \mathfrak{g}_i of $\mathfrak{V}(\mathcal{S})_1$ satisfies $h_i/k_i = 9/2$. Hence by Proposition 4.26, \mathfrak{g}_i is one of

Type	$A_{8,2}$	$B_{5,2}$	$F_{4,2}$
h^\vee	9	9	9
Dimension	80	55	52

Hence the Lie algebra structure is $A_{8,2}F_{4,2}$. \square

Case: \mathcal{S} as in Proposition 5.8 (3).

Proposition 5.11. *Let \mathcal{S} be a maximal totally singular subspace of $R(X) \oplus R(V)$. Assume that $\rho_1(\mathcal{S}^{(2)}) = \text{Span}_{\mathbb{F}_2}\{[0]^-, [\alpha_1]^+, [\alpha_{(1^4 0^{12})}/2]^+, [\alpha_{(1^2 0^2 1^2 0^{10})}/2]^+\}$ holds. Set $D = \text{Span}_{\mathbb{F}_2}\{(1^4 0^{12}), (1^2 0^2 1^2 0^{10})\}$. Then the following hold:*

- (1) $\rho_1(\mathcal{S}) \cap \{[\alpha_c/2 - \alpha_1]^\pm, [\alpha_c/2]^\pm \mid \text{wt}(c) = 2\} = \{[\alpha_c/2]^\pm, [\alpha_c/2 - \alpha_1]^\pm \mid \text{wt}(c) = 2, c \in D^\perp\}$ and its size is 192;
- (2) $\dim \mathfrak{V}(\mathcal{S})_1 = 288$, and the Lie algebra structure of $\mathfrak{V}(\mathcal{S})_1$ is $C_{10,1}B_{6,1}$.

Proof. Note that the number of codewords in D^\perp with weight 2 is $\binom{10}{2} + 3 = 48$. (1) is easily calculated by Lemmas 3.2 and 5.6 (1).

By Proposition 5.9, we have

$$\dim \mathfrak{V}(\mathcal{S})_1 = 16 \times 3 + 4 \times 12 + 0 + 8 \times 0 + 192 \times 1 = 288.$$

By Proposition 4.25, each simple component \mathfrak{g}_i of $\mathfrak{V}(\mathcal{S})_1$ satisfies $h_i/k_i = 11$. Hence by Proposition 4.26, \mathfrak{g}_i is one of the following.

Type	$A_{10,1}$	$B_{6,1}$	$C_{10,1}$
h^\vee	11	11	11
Dimension	120	78	210

Hence the Lie algebra structure is $C_{10,1}B_{6,1}$. \square

Case: \mathcal{S} as in Proposition 5.8 (4).

Proposition 5.12. *Let \mathcal{S} be a maximal totally singular subspace of $R(X) \oplus R(V)$. Assume that $\rho_1(\mathcal{S}^{(2)}) = \text{Span}_{\mathbb{F}_2}\{[0]^-, [\alpha_{(1^4 0^{12})}/2]^+, [\alpha_{(1^2 0^2 1^2 0^{10})}/2]^+, [\alpha_{((10)^4 0^8)}/2]^+\}$. Set $D = \text{Span}_{\mathbb{F}_2}\{(1^4 0^{12}), (1^2 0^2 1^2 0^{10}), (10)^4 0^8\}$. Then the following hold:*

- (1) $\rho_1(\mathcal{S}) \cap \{[\alpha_c/2 - \alpha_1]^\pm, [\alpha_c/2]^\pm \mid \text{wt}(c) = 2\} = \{[\alpha_c/2]^\pm, [\alpha_c/2 - \alpha_1]^\pm \mid \text{wt}(c) = 2, c \in D^\perp\}$ and its size is 144;
- (2) $\dim \mathfrak{V}(\mathcal{S})_1 = 216$, and the Lie algebra structure of $\mathfrak{V}(\mathcal{S})_1$ is $D_{9,2}A_{7,1}$.

Proof. Note that the number of codewords in D^\perp with weight 2 is $\binom{10}{2} + 3 = 48$. (1) is easily calculated by Lemmas 3.2 and 5.6 (1).

By Proposition 5.9, we have

$$\dim \mathfrak{V}(\mathcal{S})_1 = 16 \times 1 + 4 \times 14 + 0 + 8 \times 0 + 144 \times 1 = 216.$$

By Proposition 4.25, each simple component \mathfrak{g}_i of $\mathfrak{V}(\mathcal{S})_1$ satisfies $h_i/k_i = 8$. Hence by Proposition 4.26, \mathfrak{g}_i is one of the following.

Type	$A_{7,1}$	$C_{7,1}$	$D_{5,1}$	$D_{9,2}$
h^\vee	8	8	8	16
Dimension	63	105	45	153

Hence $\mathfrak{V}(\mathcal{S})_1$ is $(A_{7,1})^2(D_{5,1})^2$ or $D_{9,2}A_{7,1}$. Clearly, $\mathfrak{V}(\{([0]^-, [0]^+)\})_1$ is an abelian subalgebra of $\mathfrak{V}(\mathcal{S})$. By the fusion rules $[0]^- \times [\lambda]^\pm = [\lambda]^\mp$ and $\mathcal{S}^{(1)} = 0$, it is maximal abelian. By Lemma 5.5, it is toral. Hence the rank of $\mathfrak{V}(\mathcal{S})_1$ is 16. Thus the Lie algebra structure of $\mathfrak{V}(\mathcal{S})_1$ is $D_{9,2}A_{7,1}$. \square

Case: \mathcal{S} as in Proposition 5.8 (5).

Proposition 5.13. *Let \mathcal{S} be a maximal totally singular subspace of $R(X) \oplus R(V)$. Assume that $\rho_1(\mathcal{S}^{(2)}) = \text{Span}_{\mathbb{F}_2}\{[0]^-, [\alpha_{(1^4 0^{12})}/2]^+, [\alpha_{(1^2 0^2 1^2 0^{10})}/2]^+, [\alpha_{(1^{16})}/4]^+\}$. Set $D = \text{Span}_{\mathbb{F}_2}\{(1^4 0^{12}), (1^2 0^2 1^2 0^{10})\}$. Then the following hold:*

- (1) $\rho_1(\mathcal{S}) \cap \{[\alpha_c/2 - \alpha_1]^\pm, [\alpha_c/2]^\pm \mid \text{wt}(c) = 2\} = \{[\alpha_c/2 - \alpha_1]^\pm \mid \text{wt}(c) = 2, c \in D^\perp\}$ and its size is 96;
- (2) $\dim \mathfrak{V}(\mathcal{S})_1 = 144$, and the Lie algebra structure is $A_{9,2}A_{4,1}B_{3,1}$.

Proof. Note that the number of codewords in D^\perp with weight 2 is $\binom{10}{2} + 3 = 48$. (1) is easily calculated by Lemmas 3.2 and 5.6 (1).

By Proposition 5.9, we have

$$\dim \mathfrak{V}(\mathcal{S})_1 = 16 \times 1 + 4 \times 6 + 8 + 8 \times 0 + 96 \times 1 = 144.$$

By Proposition 4.25, each simple component \mathfrak{g}_i of $\mathfrak{V}(\mathcal{S})_1$ satisfies $h_i/k_i = 5$. Hence by Proposition 4.26 \mathfrak{g}_i is one of the following.

Type	$A_{4,1}$	$A_{9,2}$	$B_{3,1}$	$B_{8,3}$	$C_{4,1}$	$D_{6,2}$
h^\vee	5	10	5	15	5	10
Dimension	24	99	21	136	36	66

By the same arguments as in Proposition 5.12, $H = \mathfrak{V}(\{([0]^-, [0]^+)\})_1$ is a Cartan subalgebra of $\mathfrak{V}(\mathcal{S})_1$. We decomposes $\mathfrak{V}(\mathcal{S})_1$ into a direct sum of root spaces for H . Then

$\mathfrak{V}(\mathcal{S})_1$ has an ideal with 90-dimensional root space

$$\mathfrak{V}(\{([\alpha_c/2 - \alpha_1]^\varepsilon, W(c, \varepsilon)) \mid \varepsilon \in \{\pm\}, c \in D^\perp, \text{wt}(c) = 2, \text{supp}(c) \cap \text{supp}(D) = \emptyset\})_1,$$

where $W(c, \varepsilon)$ is a unique element in $R(V)$ such that $([\alpha_c/2 - \alpha_1]^\varepsilon, W(c, \varepsilon)) \in \mathcal{S}$. Hence the root space of $\mathfrak{V}(\mathcal{S})_1$ has the decomposition $90 + 38$, and the Lie algebra structure is $A_{9,2}A_{4,1}B_{3,1}$ or $A_{4,1}(B_{3,1})^6$ or $A_{4,1}^3(B_{3,1})^2C_{4,1}$. Since the rank of $\mathfrak{V}(\mathcal{S})_1$ is 16, it must be $A_{9,2}A_{4,1}B_{3,1}$. \square

Case: Proposition 5.8 (6).

Proposition 5.14. *Let \mathcal{S} be a maximal totally singular subspace of $R(X) \oplus R(V)$. Assume that $\rho_1(\mathcal{S}^{(2)}) = \text{Span}_{\mathbb{F}_2}\{[\alpha_{(1^4 0^{12})}/2]^+, [\alpha_{(1^2 0^2 1^2 0^{10})}/2]^+, [\alpha_{(1^{16})}/4]^+, [\chi_0]^+\}$. Set $D = \text{Span}_{\mathbb{F}_2}\{(1^4 0^{12}), (1^2 0^2 1^2 0^{10})\}$. Then the following hold:*

- (1) $\rho_1(\mathcal{S}) \cap \{[\alpha_c/2 - \alpha_1]^\pm, [\alpha_c/2]^\pm \mid \text{wt}(c) = 2\} = \{[\alpha_c/2 - \alpha_1]^+ \mid \text{wt}(c) = 2, c \in D^\perp\}$ and its size is 48;
- (2) $\dim \mathfrak{V}(\mathcal{S})_1 = 72$, and the Lie algebra structure is $D_{5,4}C_{3,2}(A_{1,1})^2$.

Proof. Note that the number of codewords in D^\perp with weight 2 is $\binom{10}{2} + 3 = 48$. (1) is easily calculated by Lemmas 3.2 and 5.6 (1).

By Proposition 5.9, we have

$$\dim \mathfrak{V}(\mathcal{S})_1 = 16 \times 0 + 4 \times 3 + 12 + 8 \times 0 + 48 \times 1 = 72.$$

By Proposition 4.25, each simple component \mathfrak{g}_i of $\mathfrak{V}(\mathcal{S})_1$ satisfies $h_i/k_i = 2$. Hence by Proposition 4.26, \mathfrak{g}_i is one of the following.

Type	$A_{1,1}$	$A_{3,2}$	$A_{5,3}$	$A_{7,4}$	$C_{3,2}$	$C_{5,3}$	$D_{4,3}$	$D_{5,4}$	$D_{6,5}$	$G_{2,2}$
h^\vee	2	4	6	8	4	6	6	8	10	4
Dimension	3	15	35	63	21	55	28	45	66	14

By Proposition 3.1, both

$$\mathfrak{V}(\{([\alpha_c/2 - \alpha_1]^+, W(c)) \mid \text{wt}(c) = 2, c \in D^\perp, \text{supp}(c) \cap \text{supp}(D) = \emptyset\})_1$$

and

$$\mathfrak{V}(\mathcal{S}^{(2)} \cup \{([\alpha_c/2 - \alpha_1]^+, W(c)) \mid \text{wt}(c) = 2, c \in D^\perp, \text{supp}(c) \subset \text{supp}(D)\})_1$$

are ideals, where $W(c)$ is a unique element in $R(V)$ such that $([\alpha_c/2 - \alpha_1]^+, W(c)) \in \mathcal{S}$. Hence $\mathfrak{V}(\mathcal{S})$ has the decomposition $45 + 27$. Moreover, by the previous case, the first subspace is the fixed points of the Lie algebra of type $A_{9,2}$ by an order 2 automorphism acting by -1 on the Cartan subalgebra. Hence $\mathfrak{V}(\mathcal{S})_1$ contains D_5 as an ideal, and the Lie algebra structure is $D_{5,4}(A_{1,1})^9$, $D_{5,4}A_{3,2}(A_{1,1})^4$ or $D_{5,4}C_{3,2}(A_{1,1})^2$. Since

$\mathfrak{V}(\text{Span}_{\mathbb{F}_2}\{[\alpha_{(1^4 0^{12})}/2]^+, [\alpha_{(1^2 0^2 1^2 0^{10})}/2]^+, [\alpha_{(1^{16})}/4]^+\})$ is isomorphic to $V_{A_3 \oplus A_4}^+$, $\mathfrak{V}(\mathcal{S})_1$ contains a Lie subalgebra $C_{2,2}(A_{1,2})^2$. Hence the Lie algebra structure is $D_{5,4}C_{3,2}(A_{1,1})^2$. \square

5.4. Isomorphism type of $\mathfrak{V}(\mathcal{S})$ for \mathcal{S} in Proposition 5.8 (4). In this subsection, we show that the VOA associated to the maximal totally singular space given in Proposition 5.8 (4) is isomorphic to $\tilde{V}_{N(A_{17}E_7)}$ as a VOA.

First, we construct a lattice VOA as a simple current extension of $X \otimes V$.

Lemma 5.15. *Let \mathcal{S} be a maximal totally singular subspace of $R(X) \oplus R(V)$ such that*

$$\rho_1(\mathcal{S}^{(2)}) = \text{Span}_{\mathbb{F}_2}\{[0]^- , [\alpha_1]^+ , [\alpha_c/2]^+ \mid c \in D\},$$

where $D = \text{Span}_{\mathbb{F}_2}\{(1^4 0^{12}), (1^2 0^2 1^2 0^{10}), ((10)^4 0^8)\}$. Then we have the following:

- (1) $\rho_1(\mathcal{S}) \cap \{[\alpha_c/2 - \alpha_1]^\pm, [\alpha_c/2]^\pm \mid \text{wt}(c) = 2\} = \{[\alpha_c/2]^\pm, [\alpha_c/2 - \alpha_1]^\pm \mid \text{wt}(c) = 2, c \in D^\perp\}$ and its size is 144;
- (2) $\mathfrak{V}(\mathcal{S})$ is isomorphic to a lattice VOA associated to $N(A_{17}E_7)$ or $N(D_{10}E_7^2)$.

Proof. Note that the number of codewords in D^\perp with weight 2 is $\binom{10}{2} + 3 = 48$. (1) is easily calculated by Lemmas 3.2 and 5.6 (1).

It follows from $\dim \rho_1(\mathcal{S}^{(2)}) = 5$ that $\dim \rho_2(\mathcal{S}^{(1)}) = 1$. Up to conjugation by $\text{Aut}(X \otimes V)$, we may assume that $\rho_2(\mathcal{S}^{(1)}) = \{[0]^+, [0]^-\}$. By Proposition 5.9, we have

$$\dim \mathfrak{V}(\mathcal{S})_1 = 16 \times 3 + 4 \times 28 + 0 + 8 \times 1 + 144 \times 2 = 456.$$

By Proposition 4.25, each simple component \mathfrak{g}_i of $\mathfrak{V}(\mathcal{S})_1$ satisfies $h_i/k_i = 18$. Hence by Proposition 4.26, \mathfrak{g}_i is one of

Type	$A_{17,1}$	$D_{10,1}$	$E_{7,1}$
h^\vee	18	18	18
Dimension	323	190	133

Hence $\mathfrak{V}(\mathcal{S})_1$ is a Lie algebra of type $A_{17,1}E_{7,1}$ or $D_{10,1}(E_{7,1})^2$. \square

Let \mathcal{S} be a maximal totally singular subspace of $R(X) \oplus R(V)$ in the lemma above and let $W = ([\alpha_{(1^{16})}/4 - \alpha_i]^+, [\chi_0]^+)$. Then $\text{Span}_{\mathbb{F}_2}\{W, \mathcal{S} \cap W^\perp\}$ satisfies Proposition 5.8 (4). Hence by the same arguments as in Proposition 4.44, $\mathfrak{V}(\text{Span}_{\mathbb{F}_2}\{W, \mathcal{S} \cap W^\perp\})$ is obtained by the \mathbb{Z}_2 -orbifold of $\mathfrak{V}(\mathcal{S})$ associated to g_W .

By the lemma above, $\mathfrak{V}(\mathcal{S})$ is isomorphic to the VOA associated to some even unimodular lattice of rank 24. Let us show that g_W is conjugate to a lift of the -1 -isometry of the lattice. By [DGH98, Appendix D], it suffices to show that g_W acts by -1 on a Cartan subalgebra of $\mathfrak{V}(\mathcal{S})_1$. Consider the subspace $\mathfrak{V}(\{([\alpha_1]^+, [0]^+), ([0]^+, [0]^-)\})_1$ of $\mathfrak{V}(\mathcal{S})_1$. Then by Lemma 5.5, it is a 24-dimensional toral abelian subalgebra, that is, a

Cartan subalgebra. Since $\{([\alpha_1]^+, [0]^+), ([0]^+, [0]^-)\} \cap W^\perp = \emptyset$, g_W acts by -1 on this Cartan subalgebra. Recall that $\tilde{V}_{N(D_{10}E_7^2)} \cong V_{N(D_5^2A_7^2)}$. Hence $\mathfrak{V}(\mathcal{S})$ must be isomorphic to $\tilde{V}_{N(A_{17}E_7)}$. Thus we obtain the following proposition.

Proposition 5.16. *The VOA $\mathfrak{V}(\mathcal{S})$ associated to a maximal totally singular subspace \mathcal{S} of $R(X) \oplus R(V)$ satisfying Proposition 5.8 (4) is isomorphic to $\tilde{V}_{N(A_{17}E_7)}$.*

Classification of the Lie algebra structures. As a summary of this section, we obtain the following theorem.

Theorem 5.17. *Let U be a holomorphic simple current extension of $V_{\sqrt{2}D_{16}^+}^+ \otimes V_{\sqrt{2}E_8}^+$. Then one of the following holds:*

- (1) *U is isomorphic to a lattice VOA V_N or its \mathbb{Z}_2 -orbifold \tilde{V}_N for some even unimodular lattice N ;*
- (2) *U contains $(V_{\sqrt{2}E_8}^+)^{\otimes 3}$ as a full subVOA;*
- (3) *The weight one space U_1 is isomorphic to one of the Lie algebras in Table 5.*

TABLE 5. Lie algebra structure of $\mathfrak{V}(\mathcal{S})_1$ for $\mathcal{S} \subset R(X) \oplus R(V)$

\mathcal{S}	$\dim \mathfrak{V}(\mathcal{S})$	Lie algebra	No. in [Sc93]	Ref.
Proposition 5.7(3)	132	$A_{8,2}F_{4,2}$	36	New
Proposition 5.8(3)	288	$C_{10,1}B_{6,1}$	56	[La11]
Proposition 5.8(4)	216	$D_{9,2}A_{7,1}$	50	$\tilde{V}_{N(A_{17}E_7)}$
Proposition 5.8(5)	144	$A_{9,2}A_{4,1}B_{3,1}$	40	[La11]
Proposition 5.8(6)	72	$D_{5,4}C_{3,2}(A_{1,1})^2$	19	[La11]

Finally, by combining Theorems 2.19, 2.20, 4.46 and 5.17, we obtain our main theorem –Theorem 0.1.

Remark 5.18. In [HS], it is announced that holomorphic framed VOAs having Lie algebra $A_{8,2}F_{4,2}$, $C_{4,2}A_{4,2}^2$ and $D_{4,4}A_{2,2}^4$ would be constructed as simple current extensions of $V_{\sqrt{2}E_8 \oplus \sqrt{2}D_4}^+ \otimes V_{\sqrt{2}D_{12}^+}^+$.

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